CPSC 536F Sept 10, 2025

- Thank to M. L. :

$$\Phi(x,y) =$$

$$(1, \sqrt{2}\,x, \sqrt{2}\,y, x^2, \sqrt{2}\,xy, y^2)$$

not

$$\cancel{(1, x, y, x^2, xy, y^2)}$$

This is positive semi-definite,

not positive definite.

In some applications, you want

a positive definite kernel function

Today, we'll finish the calculation, and show

$$k(\vec{x}, \vec{x}') = (1 + \vec{x} \cdot \vec{x}')^2$$

is positive semidefinite

WARNING: It can be misleading to write $\vec{x}$ instead of x, since

$k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ does not involve the linear structure of $\mathcal{X}$

2^{nd} aspect of kernel trick:

We use: $\vec{x}, \vec{x}' \in \mathbb{R}^n$

$(x^{test}, y^{test}) \rightsquigarrow \vec{x}^{test}$

$$= (x_1^{test}, x_2^{test}, \dots, x_n^{test})$$

Warning: In mathematics:

$$k = k(x, y) \qquad x, y \in \mathbb{R}^n$$

$$\mathcal{X}$$

$$\vdots$$

The Kernel Trick (1)

Define $k(x, x') : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$k(x, x') = (1 + \vec{x} \cdot \vec{x}')^2$$

$$= \boxed{(1 + x_1 x_1' + x_2 x_2')^2}$$

Trick Part (1)

$$\left[\vec{x} = (x_1, x_2), \ \vec{x}' = (x_1', x_2') \right]$$

$$= 1 + (x_1 x_1')^2 + (x_2 x_2')^2$$

$$+ 2(x_1 x_1' + x_2 x_2' + x_1 x_1' x_2 x_2')$$

$=$

$$(1, \sqrt{2}x_1, \sqrt{2}x_2, x_1^2, \sqrt{2}x_1x_2, x_2^2) \cdot$$

$$(1, \sqrt{2}x_1', \sqrt{2}x_2', (x_1')^2, \sqrt{2}\, x_1'x_2', (x_2')^2)$$

$=$

in $\mathbb{R}^6$

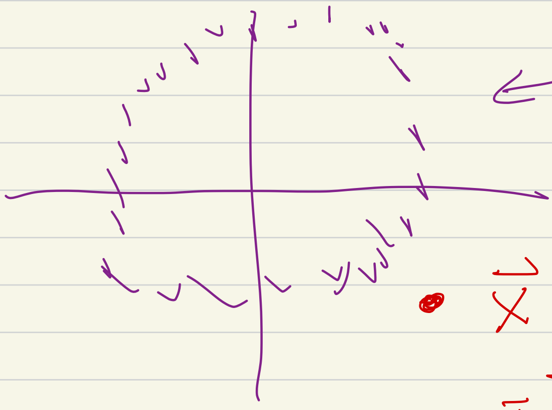
$$\Phi(x_1, x_2) \cdot \Phi(x_1', x_2')$$

$$\boxed{= \Phi(\vec{x}) \cdot \Phi(\vec{x}')}$$

where

$$\Phi(\vec{x}) = (1, \sqrt{2}x_1, \sqrt{2}x_2, x_1^2, \sqrt{2}x_1x_2, x_2^2)$$

2nd Part of Kernel trick!



$\leftarrow \{\text{cows}\} = C$

$\vec{x}^{\,test} = (x_1^{test}, x_2^{test})$

$= \vec{x}$

Say $\text{Avg}\left(\Phi(C)\right)$ in $\mathbb{R}^6$

$\approx \text{Avg}\left(\Phi\left(\begin{array}{l}\text{circle} \\ \text{radius} \\ r \\ \text{about } (0,0)\end{array}\right)\right)$

we have to see under what circumstances this holds

For now:

$$\left\| \Phi(\vec{x}^{\,test}) - \operatorname{Avg}(\Phi(C_i)) \right\|^2$$

in $\mathbb{R}^6$

$$\Phi(C) = \{ \Phi(c) \mid c \in C_i \}$$

$\subset \mathbb{R}^6$

Claim: only depends on $\Phi(\vec{x}) \cdot \Phi(\vec{x}') \ldots$