CPSC 536F Sept. 5, 2025

More on kernels:

A **kernel** on a set, $\mathcal{X}$, is a map

$$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}.$$

Warning: in mathematics, a kernel on a set, X, is a map

$$k : X \times X \to \mathbb{R}.$$

"In mathematics, you don't understand things. You just get used to them."

– John von Neumann

"In mathematics, it takes a while for things to 'sink in'."

– A. Friedman

Example: if $n \in \mathbb{N} = \{1, 2, 3, \dots\}$

$\mathcal{X} = [n] = \{1, 2, \dots, n\}$

then a kernel is a map

$$k : [n] \times [n] \longrightarrow \mathbb{R} .$$

To k we associate a matrix

$$\underline{K} = \begin{bmatrix} k(1,1) & k(1,2) & \cdots & k(1,n) \\ k(2,1) & k(2,2) & \cdots & k(2,n) \\ \vdots & & & \\ k(n,1) & k(n,2) & \cdots & k(n,n) \end{bmatrix}$$

Hence a kernel on $\mathcal{X} = [n]$

can be viewed as:

(1) $n \times n$ matrix with real entries,

(2) an element of $\mathbb{R}^{n\times n}$,

(3) " " $GL(n, \mathbb{R})$, ...

Warning: One can define kernels over $\mathbb{C}$

<u>Remark</u>: Usually positive definiteness is important to kernel applications, so typically: over $\mathbb{R}$ or over $\mathbb{C}$

A matrix $K \in \mathbb{R}^{n \times n}$ is

symmetric if

(1) $K^T = K$

~~$k(x,x') = k(x',x)$~~

positive semidefinite if

(1) For all $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$

(1) $\alpha^T K \alpha \geq 0$

(2) $\alpha \cdot (K\alpha) \geq 0$

dot product

(3)

$$[\alpha_1 \cdots \alpha_n] \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & & & \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

(3) $\sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij} \alpha_i \alpha_j \geq 0$

positive definite if

① equality in $\alpha^T K \alpha \geq 0$ holds (i.e. $\alpha^T K \alpha = 0$)

iff $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$

②

③

③'

A kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$

is:

(1) <u>symmetric</u> if $\forall x, x' \in \mathcal{X}$

$$k(x, x') = k(x', x)$$

(2) <u>positive semidefinite</u> if

$\forall \mathcal{X}' \subset \mathcal{X}$, $\mathcal{X}'$ is finite

$k|_{\mathcal{X}' \times \mathcal{X}'}$ is positive semidefinite

as a finite square matrix

(3) <u>positive definite</u> if

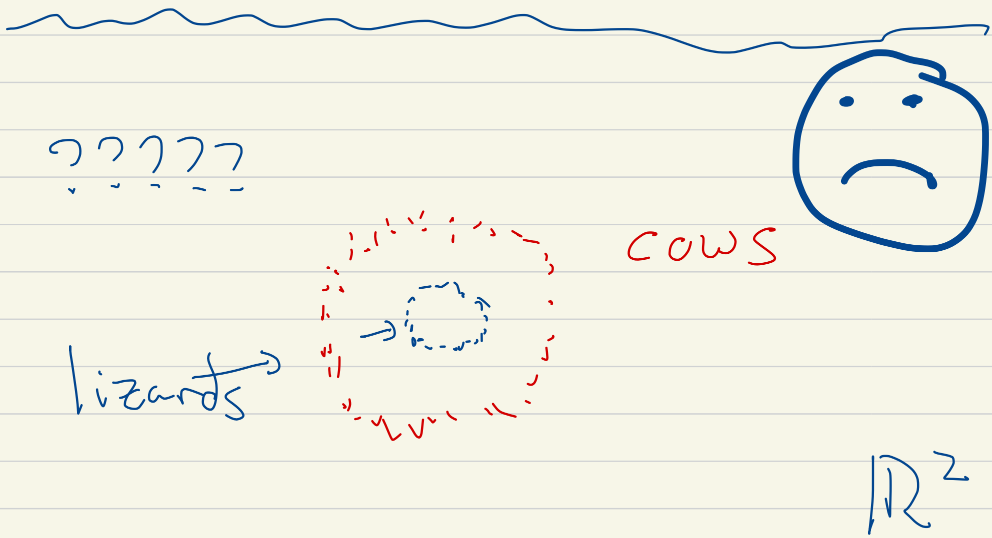
$\forall\ \mathcal{X}'' \subset \mathcal{X}\ ,\ \mathcal{X}'$ is finite

$k|_{\mathcal{X}' \times \mathcal{X}'}$ is positive ~~semi~~definite

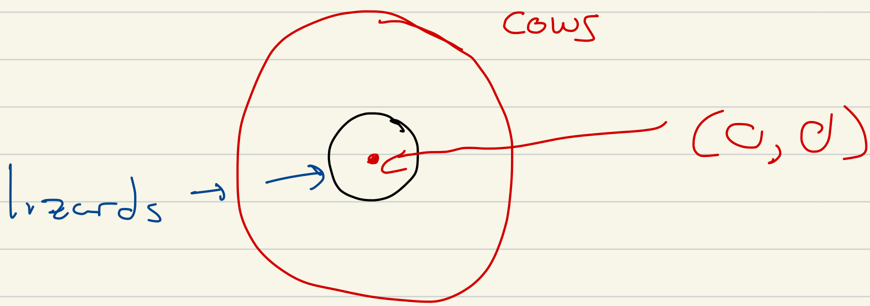
as a finite square matrix

The kernel trick

"linear separator"



Idealized mathematically:



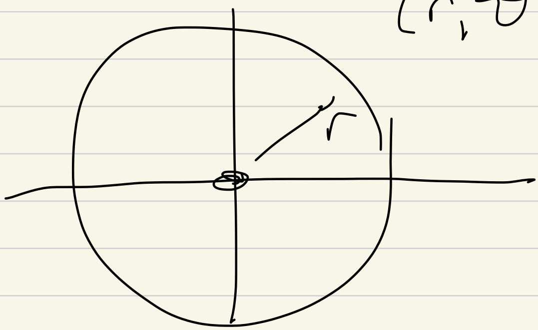
as circles about $(0,0)$

of radius r.

Trick begins:

$$\Phi(x,y) = (1, x, y, x^2, xy, y^2)$$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^6$ **(not linear!)**



(r, θ)

$0 \le \theta \le 2\pi$

"Average Value of" over circle radius r $(1, x, y, x^2, xy, y^2)$

Ans : $\left(1, 0, 0, \frac{r^2}{2}, 0, \frac{r^2}{2}\right)$

$x^2 + y^2 = r^2$

Note:

$$\text{Avg}_{\text{value}} = \left(1, c, c, \frac{r^2}{2}, c, \frac{r^2}{2}\right)$$

distinguishes between different circles

We want:

cows

$$\text{cows} \subset \mathbb{R}^2$$

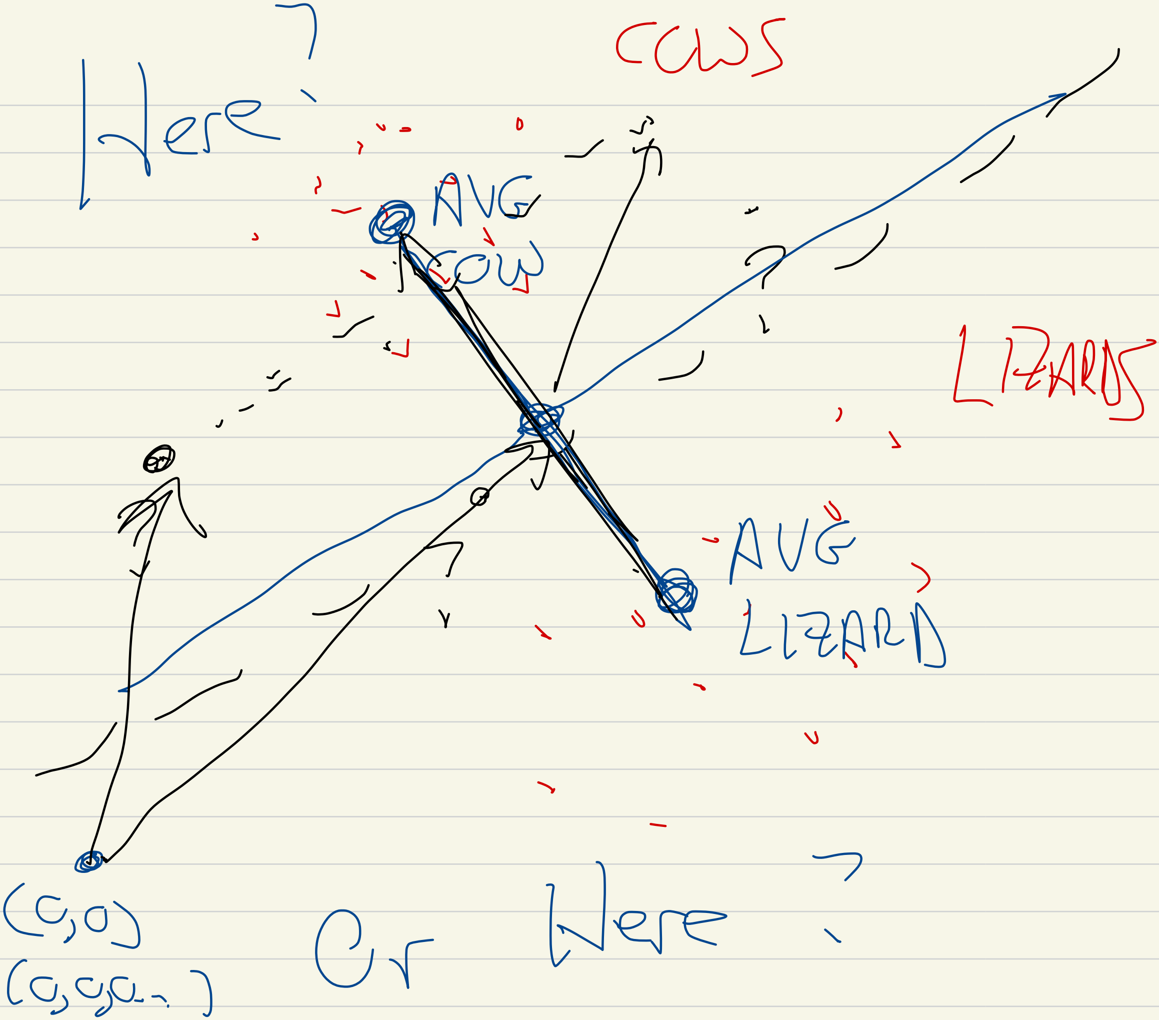
$$\text{Avg}(\text{cows}) = \frac{\sum_{(x,y)\in \text{cow}} (x,y)}{\#\,\text{cows}}$$

Avg Value $(1, x, y, x^2, xy, y^2)$

for $(x,y) \in \text{cows}$

$$= \frac{\displaystyle\sum_{(x,y)\in \text{cows}} (1, x, y, x^2, xy, y^2)}{\#\, \text{cows}}$$

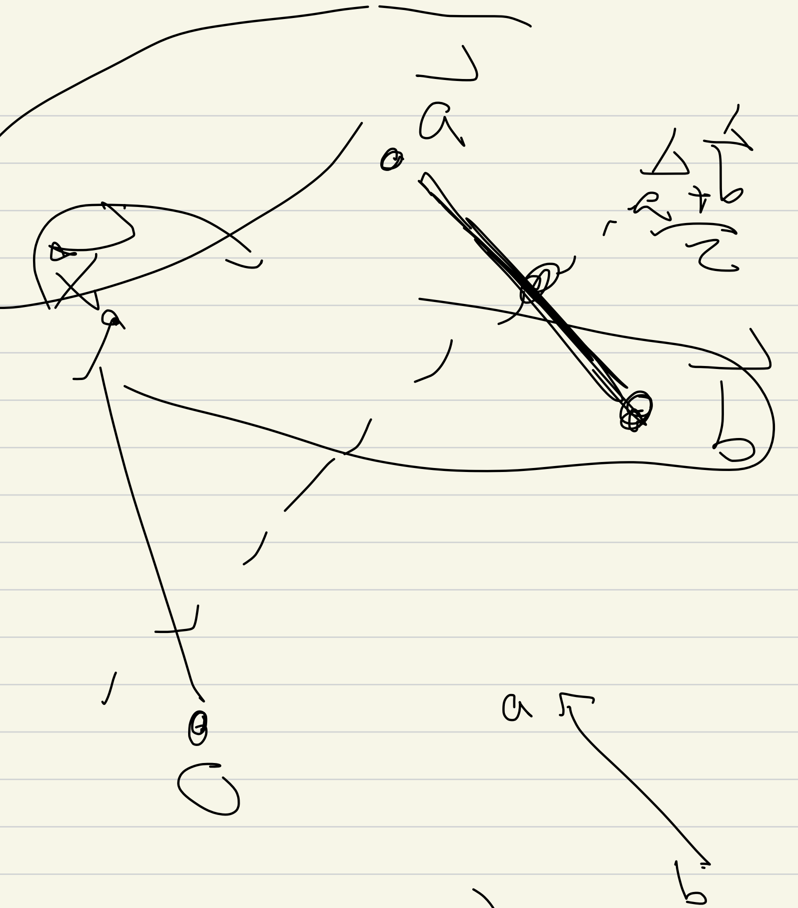
$$= \frac{\displaystyle\sum_{(x,y)\in \text{cows}} \Phi(x,y)}{\#\, \text{cows}}$$



$\vec{x}$ is a "cow"

if

$$\left(\text{AVG COW} - \text{AVG LIZARD}\right) \cdot \left(x - \frac{\text{AVG COW} + \text{AVG LIZ}}{2}\right)$$



$$\left(\vec a - \vec b\right) = ???$$

$$\|\vec x - \vec a\|^2 \gtrless \|\vec x - \vec b\|^2$$

$$\vec x \cdot \vec x - 2\vec x \cdot \vec a + \vec a^2 \gtrless \cdots$$

$\vec{x}$ closer to AVG COW than AVG LIZARD

$\Phi(\vec{x})$ " " AVG Φ(COW) than AVG Φ(LIZARD)

iff

$$\left\{ \Phi(\vec{a}) \cdot \Phi(\vec{b}), \quad \vec{a}, \vec{b} \in \text{cows} \cup (\text{LIZARDS}) \cup \{\vec{x}\} \right\}$$

then linear condition

So Φ : something $\longrightarrow \mathbb{R}^6$

$\vec{a} = (a_x ; a_y)$

$\vec{b} = (b_x , b_y)$

Then

$(1, a_x, a_y, a_x^2, a_x a_y, a_y^2) \cdot$

$(1, b_x, b_y, b_x^2, b_x b_y, b_y^2)$

$= (1 + a_x b_x + a_y b_y)^2$

$\rightsquigarrow$ never have leave $\mathbb{R}^2$

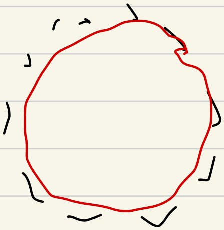
$\vec{a} \in \mathbb{R}^2 \qquad \vec{a} = (a_x, a_y)$

$\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_m \in \mathbb{R}^2$

We need to know

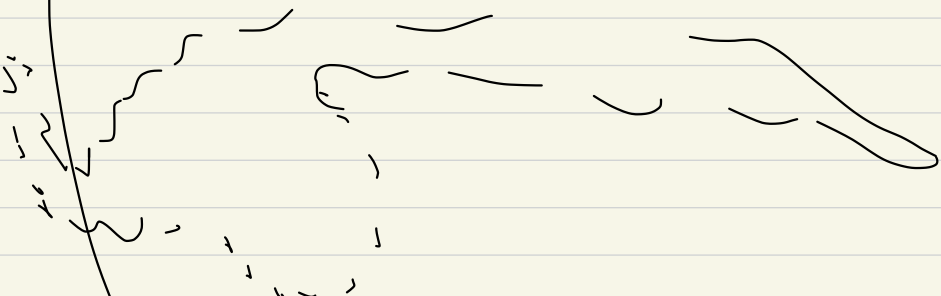
$\Phi(\vec{p}_i) \cdot \Phi(\vec{p}_j) = ??$

cows



$\subset \mathbb{R}^2$

Φ of each cow $\subset \mathbb{R}^6$



won't happen

Monday :

more examples

on Monday