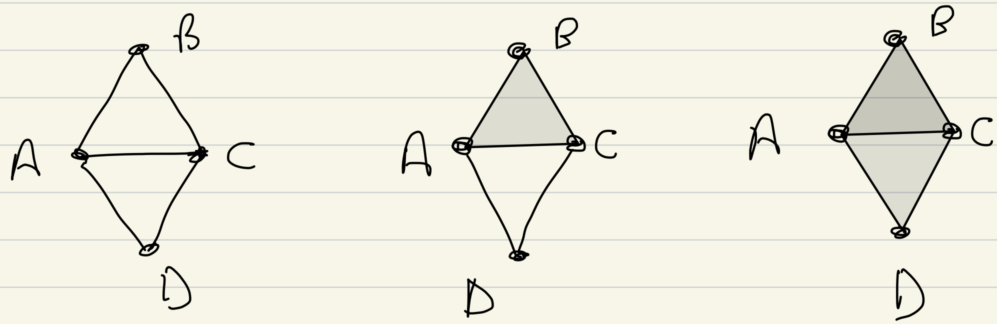
CPSC 531F Jan 17, 2025

Simplicial Complexes



K^0_{abs} K^1_{abs} K^2_{abs}

Want to compute H_0^{simp}, H_1^{simp}, H_2^{simp} of these.

One more example: Cones!

$\text{Cone}_p(K_{abs})$

If K_{abs} is an abstract simplicial complex, with vertex set $\bar{V}$,

then $K_{abs} \subset \mathrm{Power}(\bar{V})$,

$$\mathrm{Power}(\bar{V}) = \{ A \mid A \subset \bar{V} \}.$$

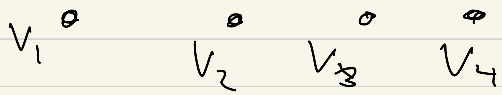
If $p \notin \bar{V}$ we define

$$\mathrm{Cone}_p(K_{abs}) = \{ A,\ A \cup \{p\} \mid A \in K_{abs} \}.$$

These turn out to be "fundamental building blocks of abs. simp. comp.

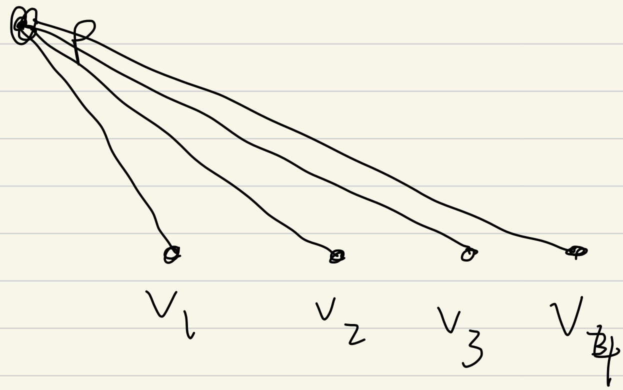
Example

K_{abs}



$\mathrm{Cone}_p(K_{abs})$

K_{abs}



write as

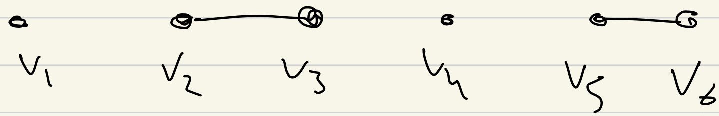
$$\langle \{p, v_1\}, \{p, v_2\}, \{p, v_3\}, \{p, v_4\} \rangle$$

For V, $S_1, \ldots, S_r \subset V$

$$\langle S_1, \ldots, S_r \rangle =$$

$$\{ A \mid A \subset S_i \text{ for some } i = 1, \ldots, r \}$$

K_{abs}

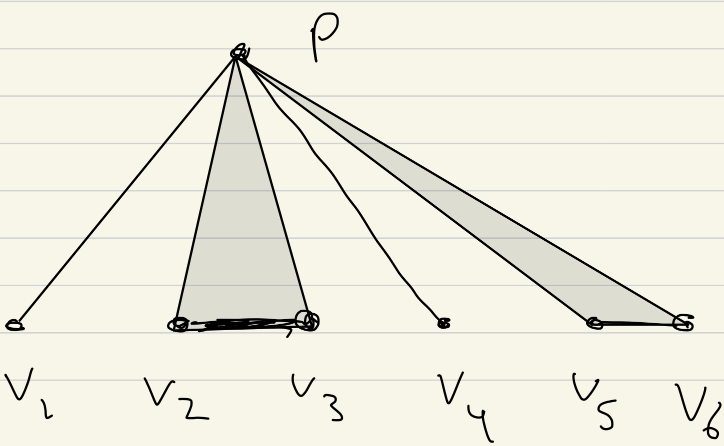


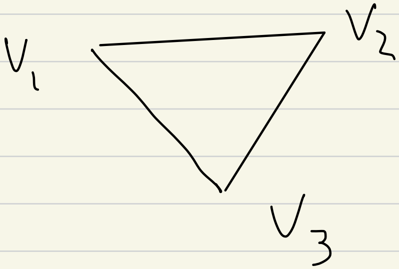
$$= \left\langle \{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5, v_6\} \right\rangle$$

$$\langle \{v_1, p\}, \{v_2, v_3, p\}, \{v_4, p\}, \{v_5, v_6, p\}$$

$\mathrm{Cone}_p(K_{abs})$

K_{abs}

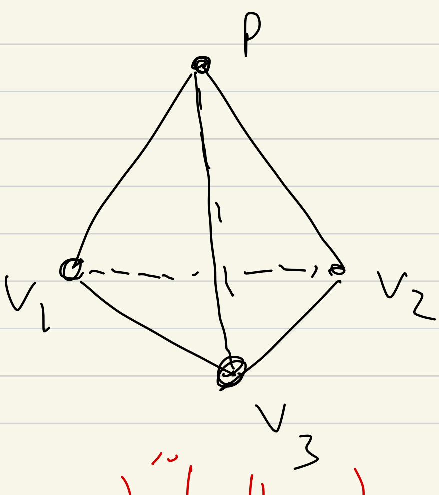




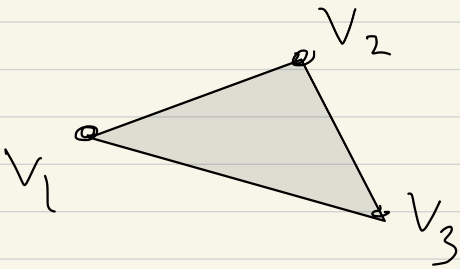
$$\left\langle \{v_1,v_2,v_3\} \right\rangle \setminus \{v_1,v_2,v_3\}$$

$$= \left\langle \{v_1,v_2\}, \{v_2,v_3\}, \{v_1,v_3\} \right\rangle$$

Cone



Trapezoid "hollowed out"

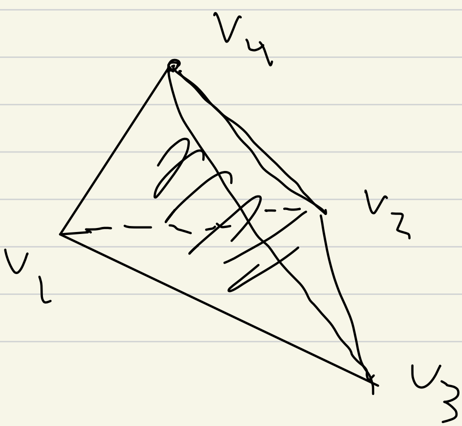


2-face,
2-dim subset
(3 elements)

$$\langle \{v_1, v_2, v_3\} \rangle$$

3-dim subset
3-face

$$\mathrm{Cone}_{v_4} \langle \{v_1, v_2, v_3\} \rangle = \langle \{v_1, v_2, v_3, v_4\} \rangle$$



Claim: For any abstract simplicial complex K_{abs}

$$H_i^{simp}\left(Cone_p(K_{abs})\right) = \begin{cases} \cong \mathbb{R} & i=0 \\ \cong 0 & i \geq 1 \end{cases}$$

Let $G = (V, E)$ be a <u>simple</u> graph

V = set,

E = collection of subsets of V of size 2.

$$\mathcal{C}_1(G) \xrightarrow{\partial_1 \,=\, \text{boundary map}} \mathcal{C}_0(G)$$

1-forms

Formal $\mathbb{R}$-linear combos of $[v, v']$ s.t.

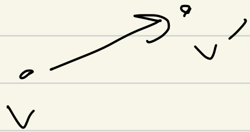
(1) $\{v, v'\} \in E$

(2) $[v, v'] = -[v', v]$

0-forms

Formal $\mathbb{R}$-linear combos of V

$$\partial_1 [v, v'] = [v'] - [v]$$



Note! $\partial_1 [v, v'] = ([v'] - [v])$ (-1)

$\qquad\quad \partial_1 [v', v] = ([v] - [v'])$

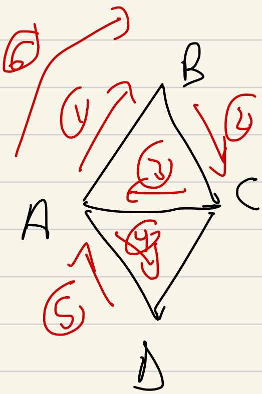
$$[v, v'] = -[v', v]$$

$$H_0(\mathcal{G})$$

$$= H_0(G) = \mathrm{coker}(\partial_1)$$

$$= C_0(G) / \mathrm{Image}(\partial_1)$$

If $G=(V,E)$ is a graph, we say that $(v_0, v_1, \ldots, v_k)$ of vertices is a <u>walk</u> in G, of length k, from v_0 to v_k if $\{v_i, v_{i+1}\} \in E$



(A, B, C, A, D, A, B)

"reverse walk"

(B, A, D, A, C, B, A)

G is <u>connected</u> if for all $v, v' \in V$, $v \neq v'$, there is a walk from v to v' in G.

To a walk

$(v_0, v_1, \ldots, v_k)$; we associate

$$[v_0, v_1] + [v_1, v_2] + \ldots + [v_{k-1}, v_k] \in C_1(G)$$

a 1-form,

$$\partial_1 \left([v_0, v_1] + [v_1, v_2] + \ldots + [v_{k-1}, v_k] \right)$$

$$= \left([v_1] - [v_0] \right) + \left([v_2] - [v_1] \right) + \ldots + \left([v_k] - [v_{k-1}] \right)$$

$$= [v_k] - [v_0] .$$

Hence, if G is connected,

for all $v, v' \in V$

$v \neq v'$, $[v'] - [v] \in \mathrm{Image}(\partial_1)$

$\partial_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$

What is

$$H_0^{simp} \stackrel{def}{=} \mathcal{C}_0 / \mathrm{Image}(\partial_1)$$

$$= \mathcal{C}_0 / B_1$$

B_1 = "~~boundaries~~ in $\mathcal{C}_0$ of ∂_1"

$= \mathrm{Image}(\partial_1)$

Recall: If $\bar{U} \subset \bar{W}$ $\mathbb{R}$-vector spaces

$$\bar{W}/\bar{U} = \left\{ w + \bar{U} \;\middle|\; w \in \bar{W} \right\},$$

where

$$w + \bar{U} = \left\{ w + u \;\middle|\; u \in \bar{U} \right\}$$

This becomes a vector space

$$(w + \bar{U}) + (w' + \bar{U}) = (w + w') + \bar{U}$$

$$\dim(\bar{W}/\bar{U})$$

$$= \dim(\bar{W}) - \dim(\bar{U})$$

We also write $w + \bar{U} = w' + \bar{U}$

as $$w \equiv_{\bar{U}} w'$$

or $$w = w' \pmod{\bar{U}}$$

If you look $\bar{W} \to \bar{W}/\bar{U}$

then w, w' are taken to

same element $w + \bar{U}$ or $w' + \bar{U}$

in $W/\bar{U}$

$H_0^{simp}(\mathcal{C})$

$= \{\text{all } 0\text{-form}\} \Big/ \left\{\begin{array}{l}\text{the } 0\text{-forms} \\ \text{that come} \\ \text{as boundaries} \\ \text{of } 1\text{-forms}\end{array}\right\}$

$(u_0, u_1, \ldots, u_k) \mapsto [u_k] - [u_0]$ in

Note!

$\partial_1 [v, v'] = 1\cdot[v'] + (-1)[v]$

Hence if

$\sum \alpha_v [v]$ that is

in the (image of ∂_1) $= B_1$, then

$$\sum \alpha_v = 0$$

i.e. $\mathbb{1} = (1, 1, \dots, 1)$

then

$$\left(\partial_1 [v, v'] \right) \cdot \mathbb{1} = 0$$

Claim: If G is connected:

(1) $\forall\, v, v' \in \bar{V}$

$$[v] + B_1 = [v'] + B_1$$

$$[v'] - [v] \in B_1$$

and

(2) for any $v_0 \in \bar{V}$

$[v_0] \notin \mathcal{B}_1$.

$\Rightarrow$ $[v_0]$ generates $H_0^{simp}(\mathcal{C})$

$$H_0^{sim}(\mathcal{C}) = \left(" \ \mathbb{1}^{\perp} \ " \right)^{\perp}$$