CPSC 531F Jan 13, 2025

- Simplicial Homology:
  - Graphs & 2-dim abstract complexes
  - $H_i^{simp}(G)$, $i=0,1$
  - Laplacians Δ_G^{vert}, Δ_G^{edge}

Admin: For now,

Homework 2025 in <u>Appendix A</u> of <u>Intro to Simplicial Homology</u> article.

Munkres: Elements of Algebraic Topology:

many good examples of simiplicial homology

For Laplacians ... (?), I don't know a good texbook ...

Abstract simplicial complex:

$\bar{V}$ = vertices, set, finite unless we say otherwise

K_{abs} is a set of subsets of $\bar{V}$

s.t. $A \subset \bar{V}$, $A \in K_{abs}$, and $A' \subset A$, then $A' \in K_{abs}$

A simple graph: $G = (\bar{V}, E)$, so $\bar{V}$ is a set, finite unless otherwise specified. E is a collection of pairs of subsets of $\bar{V}$ of size 2.

If K_{abs} is a simplicial complex,

$$\dim(K_{abs}) = \max_{A \in K_{abs}} (|A| - 1).$$

A graph, $G = (V, E) \longleftrightarrow$

abstract simplicial complex

$$\{\emptyset\} \cup \bar{V} \cup E = K_{abs}(G)$$

Simplicial homology:

If $G = (V, E)$, then a <u>0-form</u> on G is formal $\mathbb{R}$-linear combination:

$$\sum_{i=1}^{r} a_i \underbrace{[v_i]}$$

notation for 0-forms

$a_i \in \mathbb{R}$, $v_i \in V$,

$V = (A, B, C)$:

$$20 \cdot A + \pi^2 B + (-1.731) C$$

but: $20 \cdot A + 3B = 30A + 3B - 10A$

This becomes an $\mathbb{R}$-vector space

formally we mean:

$$\sum_{i=1}^{r} (\alpha_i, v_i)$$

and

$$\sum_{i=1}^{r} \underbrace{(\alpha_i, v_i)}_{\mathbb{R} \times V} = \sum_{i=1}^{r'} (\alpha_i', v_i')$$

if for all $v \in V$,

$$\sum_{v_i = v} \alpha_i = \sum_{v_i' = v} \alpha_i'$$

(we can also think of functions $\vec{V} \to \mathbb{R}$ as the same thing as a formal $\mathbb{R}$-linear sum on the set $\vec{V}$.)

A 1-form on $G = (V, E)$ is a formal sum

$$\sum_{i=1}^{r} \alpha_i [v_i, v_i']$$

s.t. $\{v_i, v_i'\} \in E$,

but we identify

$$[v_i, v_i'] \text{ with } -[v_i', v_i]$$

We use $\mathcal{C}_i(G)$ to denote the i-forms on G.

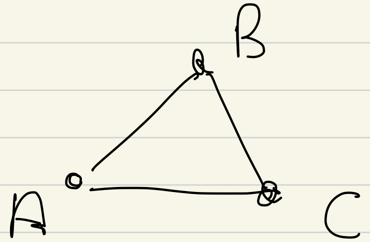
We now define

$$\partial_1 = \partial_1(G) : \mathcal{C}_1(G) \longrightarrow \mathcal{C}_0(G)$$

via

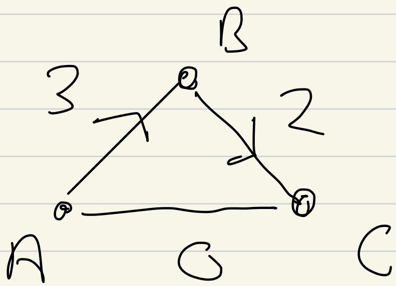
$$\partial_1\left(\sum_{i=1}^{r} \alpha_i [v_i, v_i']\right)$$

$$= \sum_{i=1}^{r} \alpha_i \left([v_i'] - [v_i] \right)$$



$$E = \{ \{A,B\}, \{A,C\}, \{B,C\} \}$$

1-form: $3\underbrace{[A,B]}_{\text{"from A to B"}} + 2[B,C]$



$$\partial_1 \left(3[A,B] + 2[B,C] \right)$$

$$= 3(B-A) + 2(C-B)$$

$$= B - 3A + 2C$$

We define: 0^{th} homology group of G

$$H_0(G) = \text{coker}(\partial_1)$$

1^{st}-homology group:

$$H_1(G) = \ker(\partial_1)$$

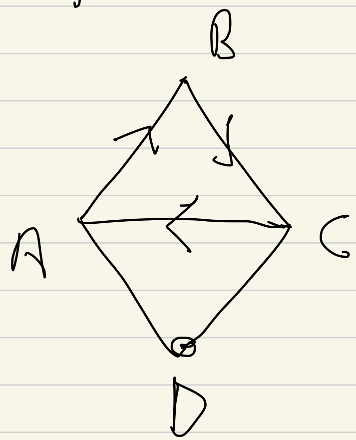
∂_1 = "boundary map"

Where

$$\ker(\partial_1) = \{ \tau \in \mathcal{C}_1(G) \mid \partial_1 \tau = 0 \}$$

$$\text{coker}(\partial_1) = \mathcal{C}_0(G) / \text{Image}(\mathcal{C}_1(G))$$

e.g.,



Claim: $H_1(G) \cong \mathbb{R}^2$

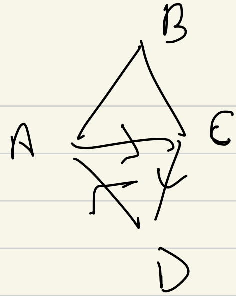
$$\tau_1 = \underbrace{[A,B] + [B,C] + [C,A]}$$

$$\partial_1(\tau_1) = \partial_1(\qquad\qquad)$$

$$= B - A + C - B + A - C = 0$$

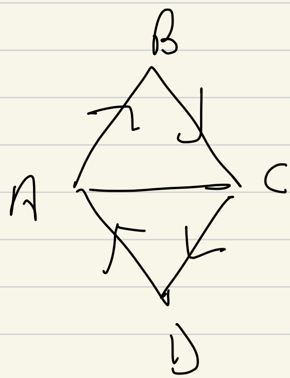
$$\tau_1 \in \ker(\partial_1)$$

$\ker(\partial_1) = Z_1(G) =$ 1-cycles in G

$\tau_2 =$ 

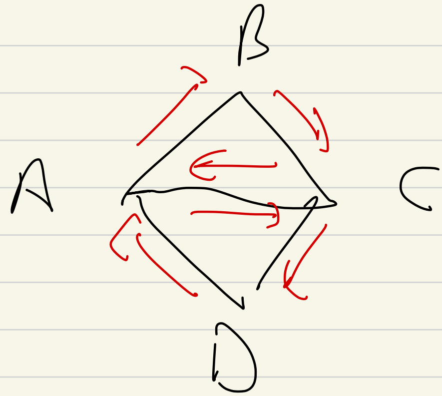
$= [A,C] + [C,D] + [D,A]$

$\partial_1(\tau_2) = C - A + D - C + A - D = 0$

$\tau_3 =$ 

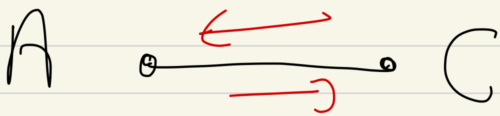
$= [A,B] + [B,C] + [C,D] + [D,A]$

$= \underline{\quad 1 \quad}\, \tau_1 + \underline{\quad 1 \quad}\, \tau_2$



intuitively:

$[C, A]$



"cancel"

$[A, C] = -[C, A]$

Theorem:

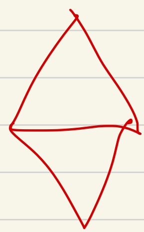
$H_1(G) = \ker(\partial_1)$

$\{\mathbb{R}\tau_1 + \mathbb{R}\tau_2\}$

We define the i-th Betti number of G to be

$$\beta_i(G) = \dim\big(H_i(G)\big)$$

Example

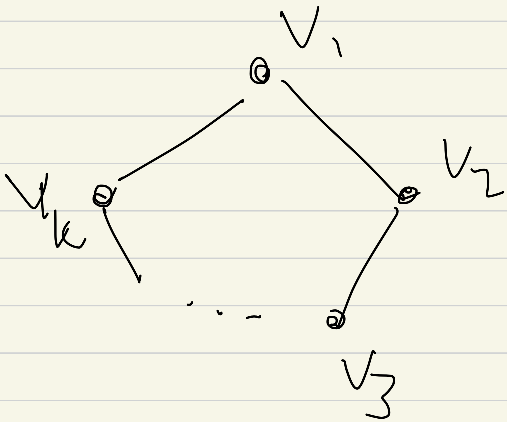
$\beta_1\Big($  $\Big) = 2$

Intuition! β_1 = "# of independent cycles in G"

Turns out

$\beta_1(G)$ = min # edges we need to remove from G so that there are no "cycles" in G

Graph theory: a cycle of length k is a graph of the form



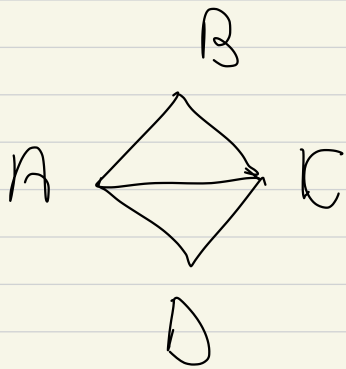
(G simple graph, need $k \geq 3$)

s.t.

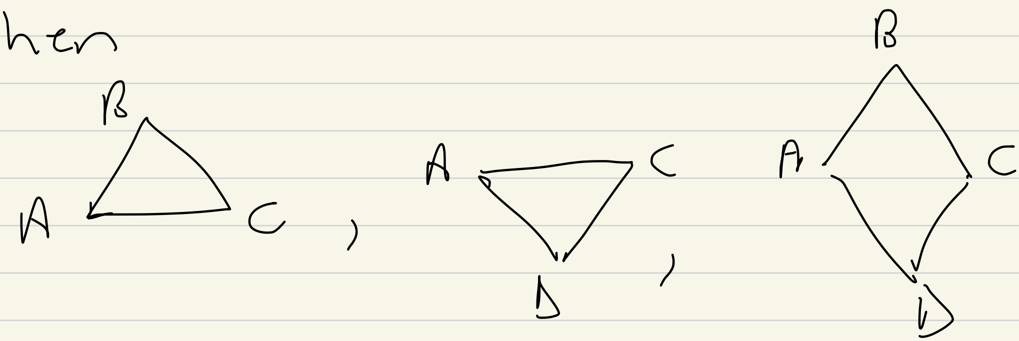
$$V = \{v_1, v_2, \ldots, v_k\}$$

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}\}$$

A <u>cycle</u> in a simple graph, G, is a subgraph $C \subset G$ s.t. C is a cycle!

$G =$ 

then



Theorem: (1) $H_0(G) \cong \mathbb{R}^{\beta_0(G)}$

$\beta_0(G) = \#$ of connected components of G

$H_0(G) = \mathcal{C}_0 / \text{Image}(\partial_1)$

(2) $\beta_0(G) - \beta_1(G)$

$= \chi(G) \stackrel{def}{=} |V| - |E|$