CPSC 531F April 8, 2021

- Next Tuesday: 3 student presentations

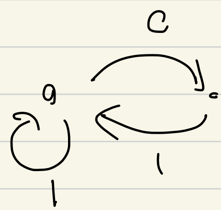
- Today:
  - (d,k) constrained data
  - Perron-Frobenius thm
    - information theory
    - Markov chains

include an optional problem or two

Please hand in final homework solutions by April 25, 11:59 pm

(d,k)-constrained data:

strings of $\{0,1\}$ s.t. between any two consecutive 1's there are between d and k 0's!

- Fibonacci 

$= G$

data 111010111010

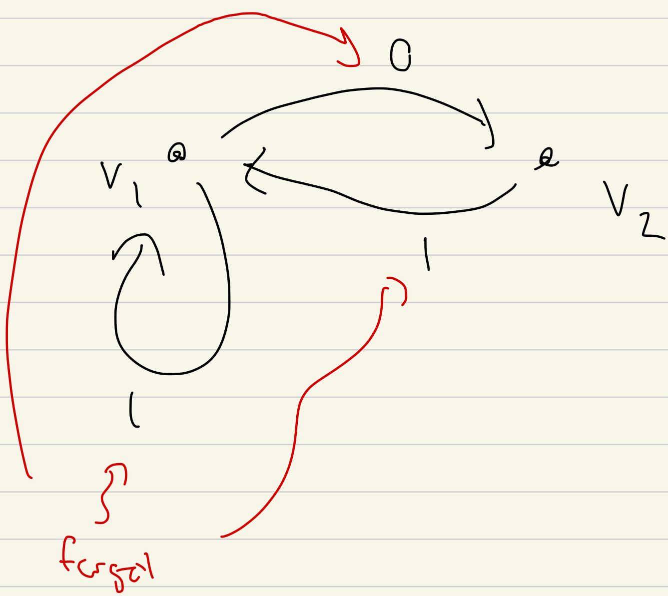
generates $(0,1)$-constr data

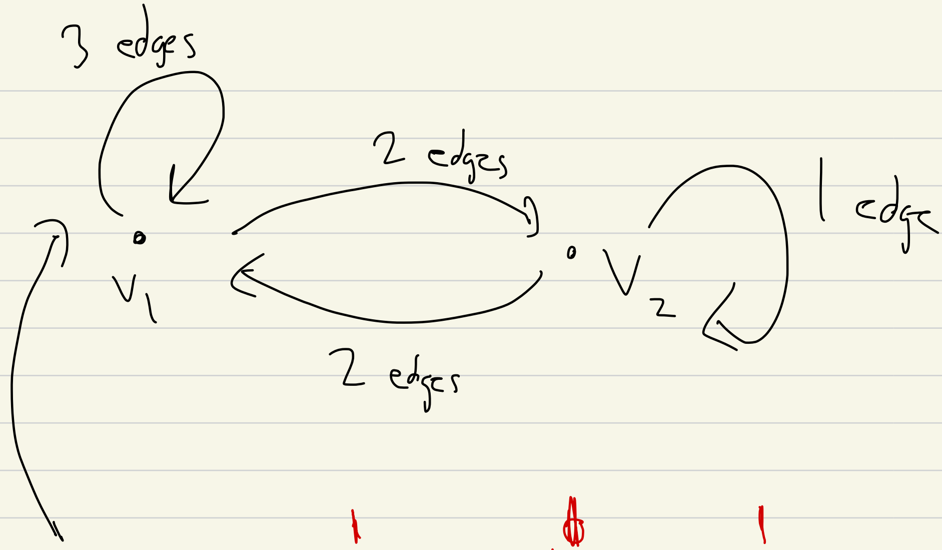
- Fib: info rate capacity, $\log_2(\lambda_{PF}) = .69\ldots > \frac{2}{3}$

$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, Fibonacci numbers

$$\underbrace{(A_G)^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}} = A_{G[3]}$$

$G[3]$: walks of length 3 on G





3 edges $\quad \overbrace{(v_1, v_1)}^{1}\ \overbrace{(v_1, v_1)}^{1}\ \overbrace{(v_1, v_1)}^{1}$

$0 \quad (v_1, v_2)\ \overbrace{(v_2, v_1)}^{1}\ \overbrace{(v_1, v_1)}^{1}$

$(v_1, v_1)\ (v_1, v_2)\ (v_2, v_1)$

capacity $G[3]$

$$= \log_2 \lambda_{pf}(G[3])$$

$$= \log_2 [\lambda_{pf}(G)]^3 = 3\,\mathrm{cap}(G)$$

$$\geq \frac{2}{3} \cdot 3 = 2$$

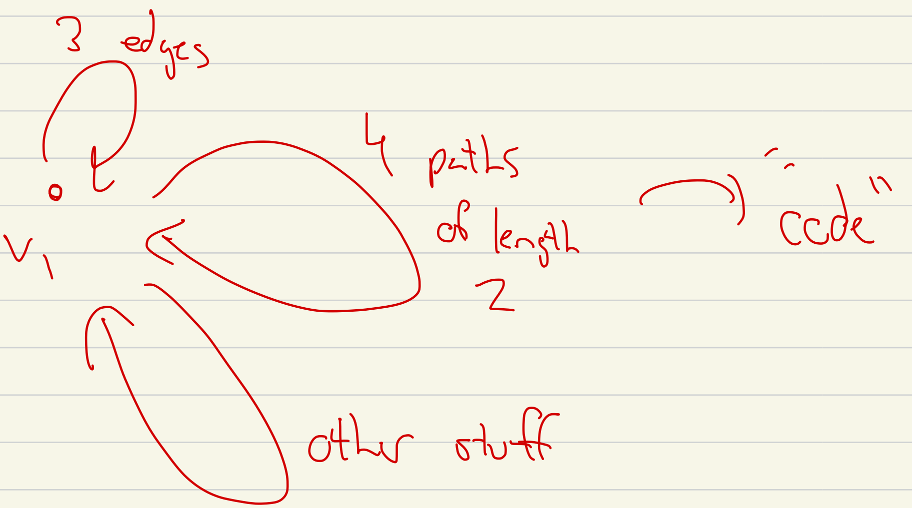
$\Rightarrow$ $G[3]$ has capacity > 2 bits

$\Rightarrow$ convert data $\{0,1\}^m$

$\longrightarrow$ data fibonacci const $\{0,1\}^{m'}$

where

$$\frac{m'}{3} \approx \frac{m}{2}$$



data:

00
01
10

3 edges in $G[3]$

11 00
" 01
" 10
" 11

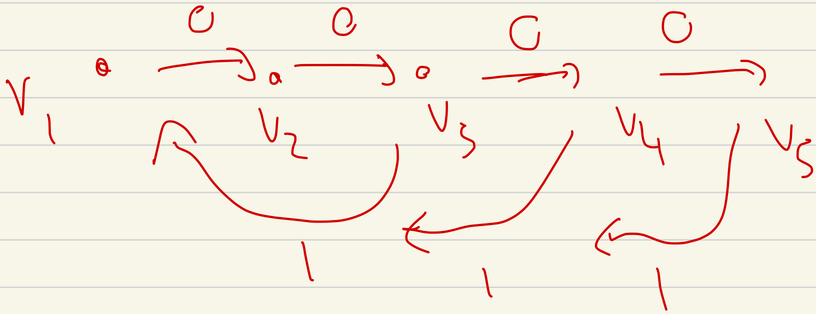
4 paths of length 2 in $G[3]$

Example: (3,3) constrained data

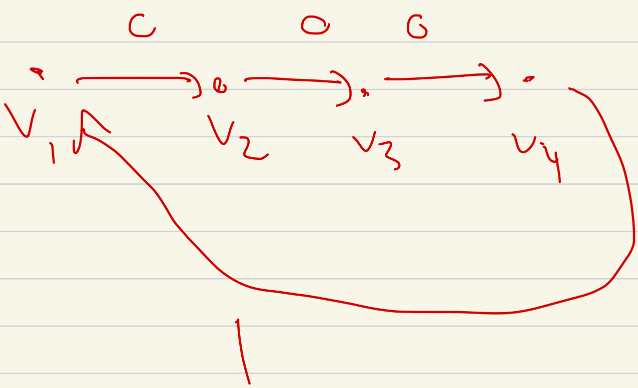
1 0 0 0 1 0 0 0 1 0 0 0 ---

0 0 1 0 0 0 1 0 0 0 1 ---

(2,4) - Constrained !

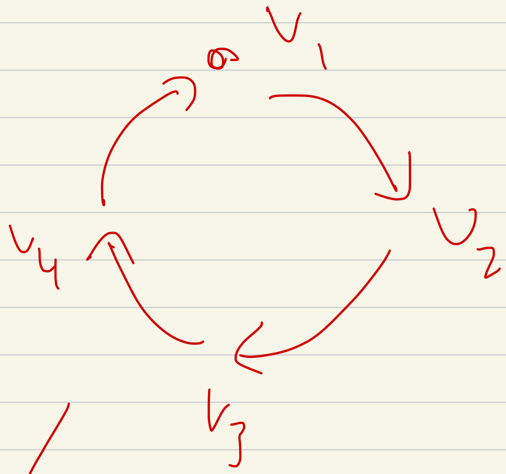


(3,3) - constrained



(3,3) constrained data

$\Longleftrightarrow$



$$A(\;) = C_4 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 1 & & & 0 \end{bmatrix}$$

λ's of C_4, $\quad 1,\ i,\ i^2,\ i^3,\quad i = \sqrt{-1}$

Power method:

$A \in M_n(\mathbb{R})$, non-neg entries,

choose v_0, $v_1 = Av_0$, $v_2 = Av_1$

$= A^2 v_0$,

... after scaling converges ... ?

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \quad \vec{v}_0, A\vec{v}_0, A^2\vec{v}_0, \ldots$$

$\vec{v}_0 \geq 0$

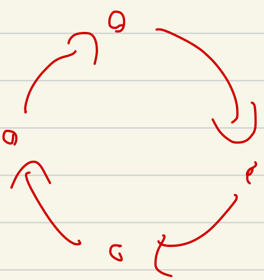
$\vec{v}_0 \neq 0$

$C_4, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$:

$v_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $C_4 v_0 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $C_4^2 v_0 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, ...

doesn't converge ---

because of periodicity

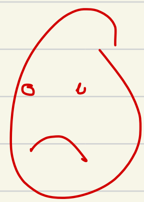


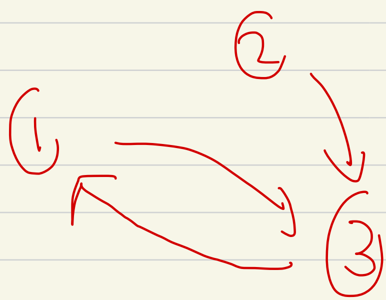
Let $A \in M_n(\mathbb{R})$ with non-neg entries.

Periodicity of A!

digraph associated to A

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 5 \\ 10 & 0 & 0 \end{bmatrix} & \\ 3 & & & \end{array}$$





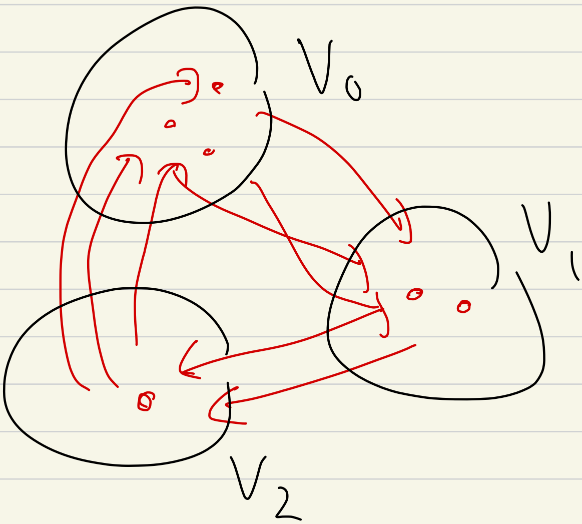
$a_{ij} > 0$

put in edge $i \to j$

Say that A is

- <u>irreducible</u> if G is strongly connected

- "period of A" = period of G



vertices of G partitioned into sets

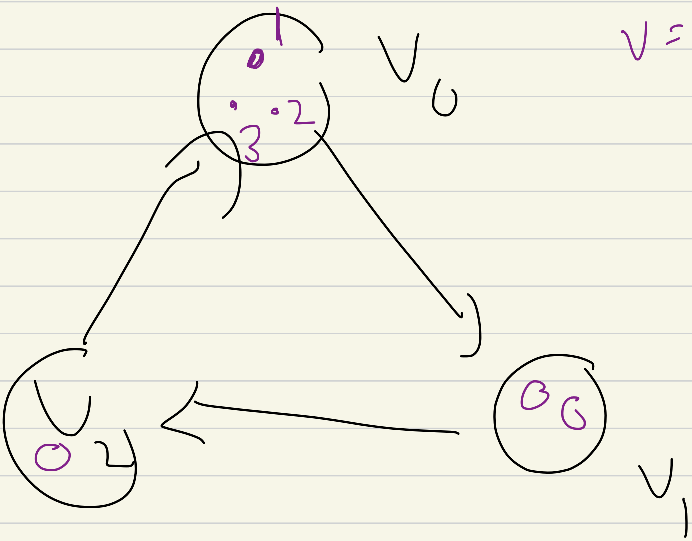
V_1, V_2, V_3

and all edges run:

from v_0 to v_1

v_1 to v_2

v_2 to v_0



$v =$

Power method won't work well

Period of G

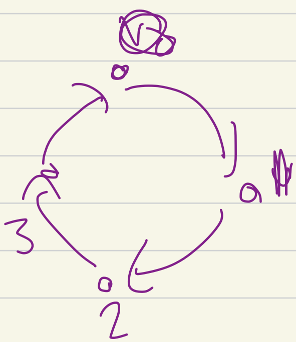
$:=$ largest $p \in \mathbb{N}$

s.t. all closed walks

$$v_0 \to v_1 \to \dots \to v_k = v_0$$

of length k have $p \mid k$.

C_4



$C_4 \iff$ irreducible Markov matrix,

and $\left(\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}\right) = \pi$

then $\pi^T C_4 = \pi^T$,

and π is the unique stochastic vector with this property

$$e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$e_0^T C_4 = (0\ 1\ 0\ 0)$$
$$e_0^T C_4^2 = [0\ 0\ 1\ 0]$$
$$\vdots$$

avg of these vectors, you get $\left(\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\right)$

Trick:

Given $A \in M_n(\mathbb{R})$, non-neg entries, looking for λ_{pf} that is "largest":

$$\boxed{\Lambda = \left\{ \lambda \;\middle|\; A\vec{v} \geq \lambda \vec{v} \text{ for some } \vec{v} \geq \vec{0} \right\}}$$

Thm: Λ has a maximum value, λ_{pf}.

(1) There is a $\vec{v} \geq \vec{0}$ s.t.

$$A\vec{v} \geq \lambda_{pf}\,\vec{v} \quad (*)$$

(2) If A is irreducible, then if $(*)$, then $A\vec{v} = \lambda_{pf}\,\vec{v}$,

(3) λ_{pf} is the largest in absolute value,

(4) $\vec{v}$ as above $\vec{v} > \vec{0}$,

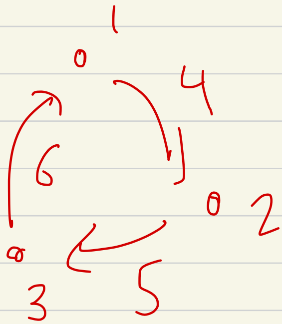
i.e. $v_i > 0$ for all $i \in [n]$

(5) λ_{pf} has multiplicity one

(6) If period of A is 1 then any other eigenvalue, λ, of A has $|\lambda| < \lambda_{pf}$; if period of A is p, then $\zeta\lambda_{pf}$ is an eigenvalue of multiplicity one for $\zeta^p = 1$, any other eigenvalue λ has $|\lambda| < \lambda_{pf}$.

Example:

$A_{3\times3}$



$$A = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 5 \\ 6 & 0 & 0 \end{bmatrix}$$

$\max \lambda$ s.t. $A\vec{v} = \lambda \vec{v}$

for some $\vec{v} \geq 0$

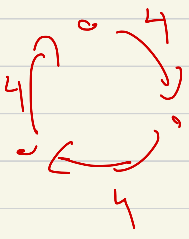
$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cancel{\geq} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \geq \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

λ has to be ≤ 0

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cancel{\geq} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \geq \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

λ is at most 4

B



$$B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \geq \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

λ can be 4

$$A^3 = \begin{bmatrix} 4\cdot5\cdot6 & 0 & 0 \\ 0 & 4\cdot5\cdot6 & 0 \\ 0 & 0 & 4\cdot5\cdot6 \end{bmatrix}$$

λ's of $\uparrow$ $4\cdot5\cdot6$ 3 times

Exercise:

$$\lambda\text{'s of } A = \sqrt[3]{4\cdot5\cdot6} \cdot \begin{Bmatrix} 1 \\ \omega \\ \omega^2 \end{Bmatrix}$$

where ω = 3rd root of unity, $\omega^3 = 1$

$\omega \neq 1$,

$$\lambda_{PF} = \sqrt[3]{4\cdot5\cdot6},$$

If you had

$$\begin{bmatrix} & 4 & \\ & & 5 \\ 6 & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

claim: if

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ \zeta y \\ \zeta^2 z \end{bmatrix}$$

with $\zeta^3 = 1$

$$\begin{bmatrix} & 4 & \\ & & 5 \\ 6 & & \end{bmatrix}\begin{bmatrix} x \\ \zeta y \\ \zeta^2 z \end{bmatrix} = \begin{bmatrix} 4\zeta y \\ 5\zeta^2 z \\ 6x \end{bmatrix} = \zeta\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

How does

$$A\vec{v} = \lambda_{PF}\vec{v}$$

work?

Claim: $\boxed{A\vec{v} = \lambda\vec{v}}$ $\quad \vec{v} \geq \vec{0}$, $\vec{v} \neq \vec{0}$

look at all λs:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

y is $\geq x, z$

$A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ middle component

$= 4x + 5y + 6z$

y largest $\leq (4+5+6)\,y.$

So

$\lambda \leq$ maximum row sum of A

$$\Lambda = \left\{ \lambda \;\middle|\; A\vec{v} \geq \lambda\vec{v} \text{ some } \begin{matrix} \vec{v} \geq \vec{0} \\ \vec{v} \neq \vec{0} \end{matrix} \right\}$$

is bounded.

Take

$$\lambda_1, \lambda_2, \ldots \in \Lambda$$

s.t.

$$\lim_{m\to\infty} \lambda_m = \lambda$$

and $\lambda \geq$ any element of Λ

So

$$A\vec{v}_m = \lambda_m \vec{v}_m$$

and, say $\|\vec{v}_m\|_2 = 1$ since

doesn't care about scaling,

So $\vec{v}_m \xrightarrow{m\to\infty} \vec{v}$

Since $\vec{v}_m \geq \vec{0}$, $\vec{v} \geq 0$

$\|\vec{v}_m\| = 1$, also $\|\vec{v}\| = 1$.

So then we have

$$A\vec{v} \geq \left(\lim_{m\to\infty} \lambda_m\right)\vec{v}$$

$$\cancel{\geq} \quad \lambda_{PF}\,\vec{v}$$

Now we claim!

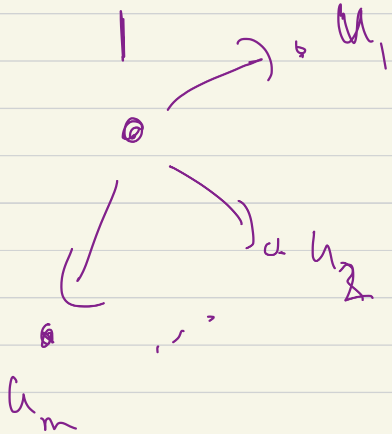
$$A\vec{v} = \lambda_{pf}\,\vec{v}$$

i.e. we can't have

$$(A\vec{v})_i > \lambda_{pf}\, v_i .$$

Otherwise 7

$$A\vec{v} \geq \lambda_{pf}\,\vec{v} \quad \text{but} \quad (A\vec{v})_i > \lambda_{pf}\, v_i$$



small $\varepsilon > 0$

$$v_1 \to v_1 + \varepsilon$$

$$(A\vec{v})_1 > \lambda_{pf} v_1$$

we can find $\varepsilon > 0$ s.t.

$$A(\vec{v} + \varepsilon \vec{e}_1) > \lambda_{pf}(\vec{v} + \varepsilon \vec{e}_1)$$

repeat this idea

To be continued $\left\{ \begin{array}{l} \text{in notes} \\ \\ \text{next time} \\ \text{I teach this} \end{array} \right.$

Class ends

11:15 Matteo

Basis vector space V:

$\{v_i\}_{i \in I}$ s.t. any element of V is a finite lin combo of $\{v_i\}_{i \in I}$

$$(Lv, w)_{\bar{W}} = (v, L^* w)_{\bar{V}}$$

if

$$(Lv_i, w_j)_{\bar{W}} = (v_i, L^* w_j)_{\bar{V}}$$

for all i, j

$$L \leftrightarrow \{a_{ij}\} \qquad L^* \leftrightarrow \{a_{ji}\}$$

$\vec{V}$ infinite dim!

$(\vec{V}^*)^*$ larger than $\vec{V}$

Inner prod spaces, $\dim(V) < \infty$.

— Hilbert space!

- $\vec{V}$ vector space
- inner product
- $\vec{V}$ is <u>complete</u> under

$$\|\vec{v}\| = \sqrt{(\vec{v},\vec{v})}$$

If

$$V^* = \{ \text{bounded linear functionals} \}$$

then $(V^*)^* = V$

$$V = \mathbb{R}^{\oplus \mathbb{Z}} = \operatorname{span}\{ \ldots \vec{e}_{-1}, \vec{e}_0, \vec{e}_1, \ldots \}$$

= finite linear combo of $\vec{e}_i$

=

$l : V \to \mathbb{R}$: $l(e_i) = c_i$

and $\ldots c_{-1}, c_0, c_1, \ldots$

$$\left(\mathbb{R}^{\oplus \mathbb{Z}}\right)^* = \prod_{i \in \mathbb{Z}} \mathbb{R}$$

$$(\ldots, c_{-1}, c_0, c_1, c_2, \ldots)$$

all $c_i = 3$

$$\left(\bigoplus_{i \in \mathbb{Z}} \mathbb{R}\right)^* = \left(\prod_{i \in \mathbb{Z}} \mathbb{R}\right)$$

$$\left(\qquad \right)^{**} = \left(\prod_{i \in \mathbb{Z}} \mathbb{R}\right)^*$$ much larger than

$$(Lv, w)_{\bar{W}} = (v, L^* w)_{\bar{V}}$$

$$L : V \to W$$

$$v \mapsto Lv$$

enough to $v_1, \dots, v_n$ basis for $\bar{V}$

$w_1, \dots, w_m$ basis of $\bar{W}$

$\Rightarrow$

Fix j: $L^* w_j$ needs to be the thing in $\bar{V}$

s.t.

$$\boxed{(L^* w_j, v_i)_{\bar{V}} = (Lv_i, w_j)_{\bar{W}}}$$

$= ?$

So can you find, for any v_i

$(\,?\,, v_i)_{\bar V}$ = what it is supposed by $(Lv_i, w_j)_{\bar W}$

given

$(\,?\,, v_i)_{\bar V}$ = given for each $i = 1, \ldots, n$

does this determine ?

If

$? = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$

If $\vec{v}_1, \dots, \vec{v}_n$ ON basis!

$$(c_1\vec{v}_1 + \dots + c_n\vec{v}_n, \ \vec{v}_i)$$

$$\|$$

$$c_i$$

$$c_1(\vec{v}_1, \vec{v}_1) + c_2(\vec{v}_2, \vec{v}_1) + \dots \overset{0}{=} \text{given}$$

$$\dots \quad \vec{v}_2 \qquad \vec{v}_2 \overset{0}{=} \text{given}$$

$$, \ 0$$

$$\vdots$$

$$0$$

homogeneous!

homogeneous eq.

What if

$$c_1(\vec{v}_1, \vec{v}_1) + c_2(\vec{v}_2, \vec{v}_1) + \ldots = 0$$

$$\vec{v}_2 \qquad\qquad \vec{v}_2 \qquad = 0$$

$$\vdots$$

has nontrivial solution

$c_1, c_2, \ldots, c_n$

not all zero.

$\Rightarrow$ claim $(\ ,\ )_V$ isn't

really a dot product.

Define! in $\mathbb{R}^3$

$$(\vec{u}, \vec{v}) = 3u_1v_1 + 5u_2v_2 + 0\,u_3v_3$$

$$(\vec{e}_3, \vec{e}_3) = 0$$ can happen

bilinear, but

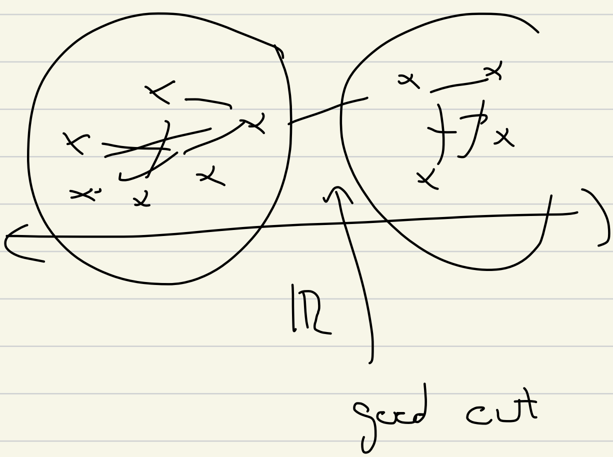
not an inner product

Graph: A_G, Δ_G

eigenvectors of $\Delta_G = D_G - A_G$

$$\Delta_G \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 \quad , \quad \lambda = 0$$

next smallest gives you



If G is d-regular

then $$\Delta_G = d \cdot I - A_G$$

$$A_G$$

$$-d \le \lambda_n \le \cdots \le \lambda_2 \le \lambda_1 = d$$

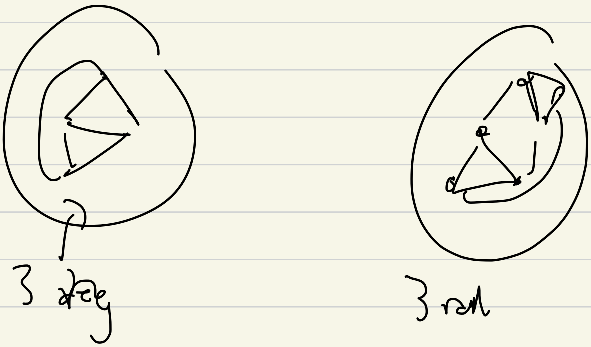
eigenvalues of Δ_G

$$2d \ge d - \lambda_n \ \cdots \ \ge d - \lambda_2 \ge 0$$

$$A_G \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = d \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \iff \Delta_G \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

If $\lambda_2(A_G) = d$

$d=3$



G

mult of d = # conn comp

" " 0 in Δ_G = "

even if not d-reg

$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ $\begin{pmatrix} a \\ a \\ a \\ -b \\ -b \\ -b \end{pmatrix}$

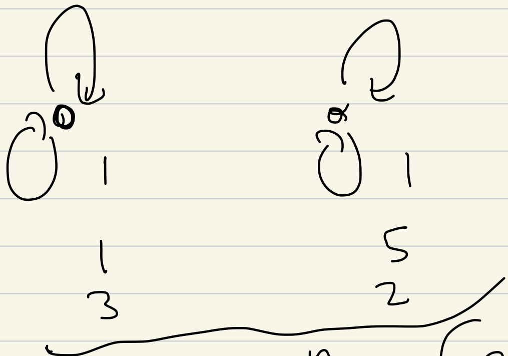
$$R_{\Delta_G}(\vec{w}) = \frac{(\Delta_G \vec{w}, \vec{w})}{(\vec{w}, \vec{w})}$$

$$= \frac{\sum_{\text{edges } e} \left(w_{t(e)} - w_{h(e)}\right)^2}{\sum_{v \text{ vertices}} w_v^2}$$

$$\vec{w} = \vec{1}$$

$$R_{\Delta_G}(\vec{1}) = 0$$

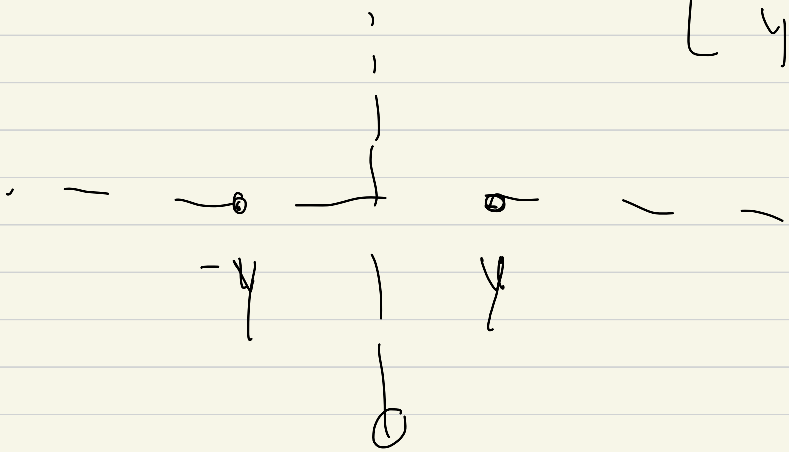
minimize $R_{\Delta_G}(\vec{w})$ s.t. $\vec{w} \perp \vec{1}$

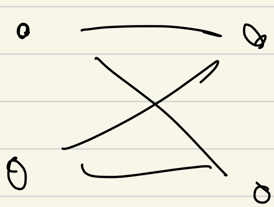


$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

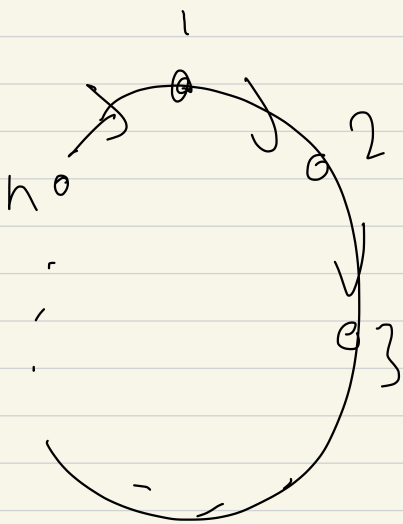
$$\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \quad y = -x$$

$$\begin{bmatrix} -y \\ y \end{bmatrix}$$



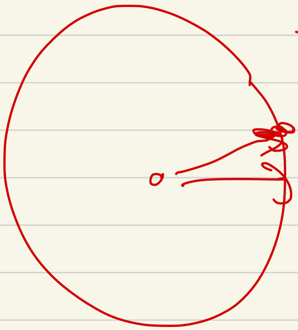


$$\left[\begin{array}{c|c} 0 & \begin{matrix} Z_2 \\ Z_2 \end{matrix} \\ \hline \begin{matrix} Z_2 \\ Z_2 \end{matrix} & 0 \end{array}\right]$$



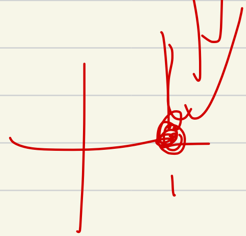
eigenvectors

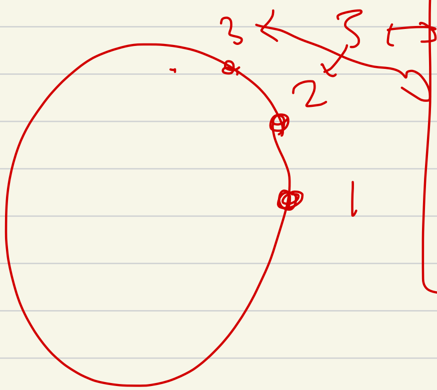
$$\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^n \end{bmatrix}$$



$\zeta = e^{\frac{2\pi i}{n}}$, close to 1

$\zeta = 1 \mapsto \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

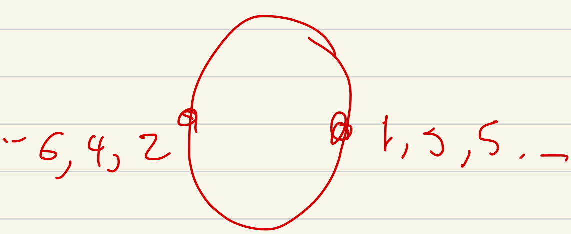




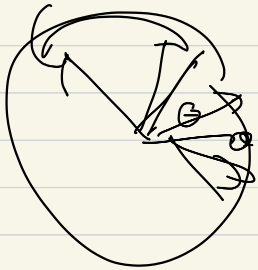
$$\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$$

n even

$$\zeta = -1 = \left(e^{\frac{2\pi i}{n}}\right)^{n/2}$$

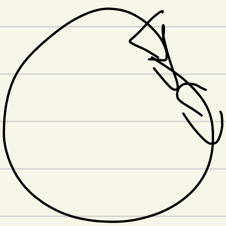


$$\zeta + \bar{\zeta} = 2\cos(\ )$$



$\leftarrow 2 \quad \zeta + \bar{\zeta}$

$$e^{2\pi i \vartheta} = \cos 2\pi\vartheta + i \sin 2\pi\vartheta$$



$C_n + C_n^{-1}$

$$\lambda = \zeta + \bar{\zeta} = 2\cos\frac{2\pi}{n}\cdot m$$