April 6, 2021 CPSC 531F

- Deadline for HW submission:
  Sunday, April 25, 11:59 pm
  (earlier submissions would be appreciated)
- Perron-Frobenius theorem!
  in the context of information theory, and Markov chains

=

Last class in next Tuesday.

Information Theory (Shannon)!

Say you want encode binary data, $\{0,1\}^{n_1}$ into $\{0,1\}^{n_2}$

data with some constraints.

E.g. (d,k)-run length constrained data:

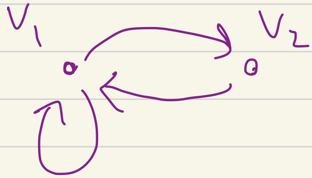
between any two consecutive 1's

there are $\geq d$ 0's

$\leq k$ 0's

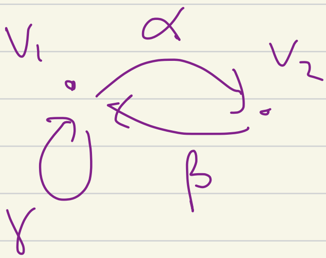
E.g.:

Fibonacci graph



$$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

this graph can generate strings:



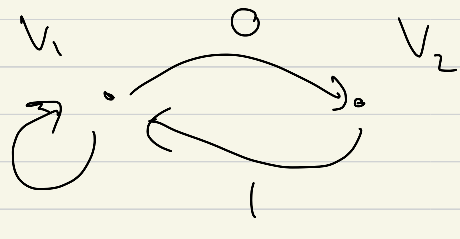
walking from v_1:

$\gamma\gamma\gamma\alpha\beta\alpha\beta$

$\gamma\alpha\beta$

$\gamma\gamma\gamma\alpha\beta\gamma\alpha$

fibonacci graph !



gives strings, start walking at V_1

$V_1 \quad V_1 \quad V_2 \quad V_1 \quad V_1$

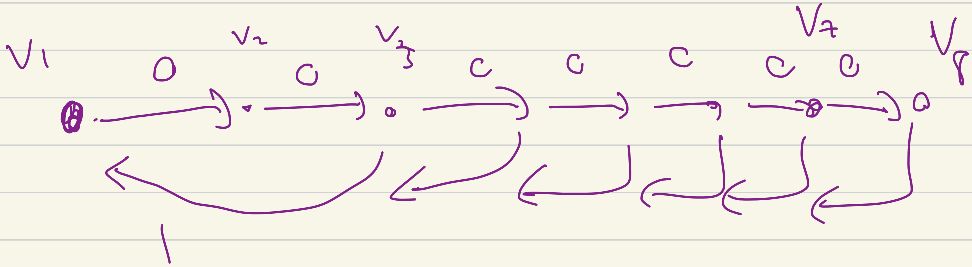
$\quad 1 \quad 0 \quad 1 \quad 1$

are (0, 1) run length constrained

between consecutive 1's have

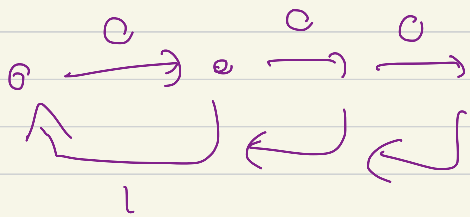
either 0, 1 occurrences of 0's

(2,7)-constrained



$$A_G = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ 1 & & 0 & 1 & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}$$

(1,3)-constrained



(2,7) graph!

walking at v_1: 0 0 1 0 0 0 0 1 0 0 1 ...

start with 00

walking at v_3: can begin

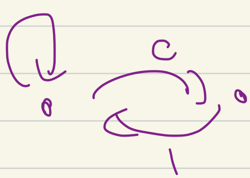
v_3		v_1		v_2		v_3		v_4		v_5		v_1	
	1		0		0		0		0		1		...

walks length n from v_1 (or v_2, ...) $\leq$ # of $\{0,1\}^n$ strings $\leq$ # walks of length n starting from anywhere

Claim!

\# $\{0,1\}^n$ strings with $(2,7)$-constraint $\sim \lambda_1(A_G)^n$

\# of $\{0,1\}^n$ strings with $(0,1)$-constraint $\sim \lambda_1\left(A_{\text{Fibonacci}}\right)^n$

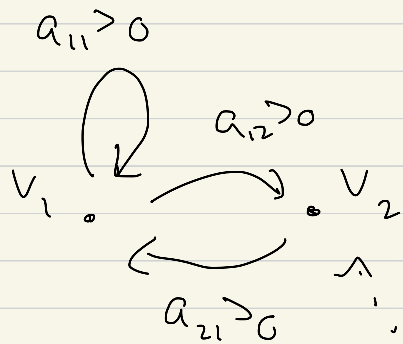


$$A_{\text{Fib}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Based on: Perron-Frob theorem:

let $A \in M_n(\mathbb{R})$ have non-neg entries, and let A be <u>irreducible</u>, i.e. the digraph associated to A is strongly connected

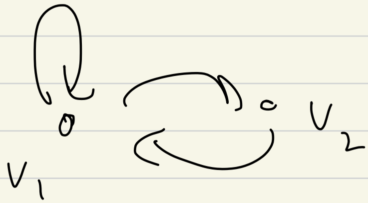
$$\begin{bmatrix} .5 & .3 \\ 700 & 0 \end{bmatrix}$$



no self-loop since $a_{21} = 0$

(A digraph is strongly connected if for each v_1, v_2 vertices, there's a path from v_1 to v_2.)

Example:



$$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

idea:

$\lambda_{pf} = \lambda_{\text{Perron-Frobenius}}$!

$$\max \left\{ \lambda \text{ s.t. } \exists\, \vec{v} \geq 0,\ \vec{v} \neq \vec{0},\ \underbrace{A\vec{v} \geq \lambda \vec{v}}_{\text{A stretches each comp of } \vec{v} \text{, by } \lambda} \right\}$$

(least upper bound)

(supremum)

Often find λ_{PF} by "power method"

$$\begin{bmatrix}1\\1\end{bmatrix}, \quad A_G\begin{bmatrix}1\\1\end{bmatrix} =$$

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix},$$

$$A_G^2\begin{bmatrix}1\\1\end{bmatrix} = A_G\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

$$A_G^3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$$

$$A_G^4 = A_G\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}8\\5\end{bmatrix}$$

$$A_G^n\begin{bmatrix}1\\1\end{bmatrix} : \underset{n=0}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{n=1}{\begin{bmatrix}2\\1\end{bmatrix}}, \begin{bmatrix}5\\3\end{bmatrix}, \begin{bmatrix}8\\5\end{bmatrix}, \begin{bmatrix}13\\8\end{bmatrix}, \ldots$$

"Power method" to find largest eigenvalue of $A \in M_n(\mathbb{C})$:

pick at "random" $v_0 \in \mathbb{C}^n$

$$v_0, Av_0, A^2 v_0, \ldots \xrightarrow{\text{normalize}} \text{largest eigenvector}$$

(this often works)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \ldots$$

$$A\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \approx \lambda \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

stretching

$$\lambda = \min\left(\frac{8}{5}, \frac{5}{3}\right)$$

$$\begin{pmatrix}5\\3\end{pmatrix} \longrightarrow A_{Fib}^{n}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{get}$$

$$\min\left(\,-\,,\,-\,\right)$$

ratios of consecutive Fib numbers

$$F_n \sim c\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$\longrightarrow \left(\frac{1+\sqrt{5}}{2}\right)^n$$

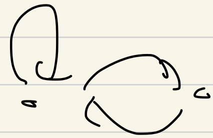
$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \overset{\text{large } n}{\approx} \begin{bmatrix} F_n \cdot \left(\frac{1+\sqrt{5}}{2}\right) \\ F_n \end{bmatrix}$$

$$A_{Fib} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \left(\frac{1+\sqrt{5}}{2}\right) \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

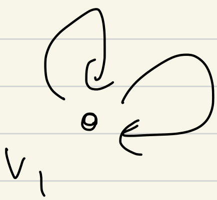
largest eigenvalue of A_{Fib}.

Define: The information capacity of G digraph

$\log_2(\lambda_{PF}(G))$ "bits" (G strongly connected)

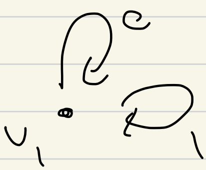
Capacity of 

$$:= \log_2\left(\frac{1+\sqrt{5}}{2}\right) = \ldots$$



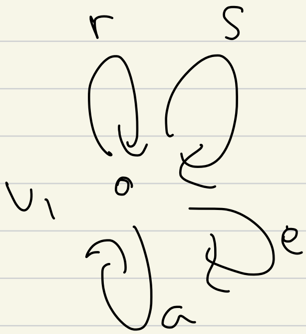
$$A_G = [2] \qquad \lambda_{PF} = 2$$

Capacity: $\log_2 2 = 1$ bit



walks of length n

$\longleftrightarrow$ $\{0,1\}$ strings of length n



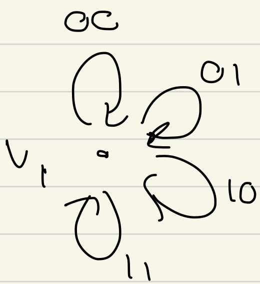
$A_G = [4]$, $\lambda_{pf} = 4$

Capacity: $\log_2 \lambda_{pf} = \log_2 4 = 2$

bits

\# walks of length $n \approx 4^n$

rsaersaceerrs ...



produces

\# walks of length n $\approx 4^n = 2^{2n}$

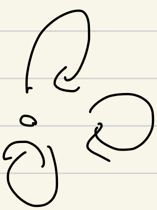
labelling gives $\xleftrightarrow{1-1}$ binary strings of length $2n$

$=$

$A_G = [8]$,

capacity $= \log_2 8 = 3$

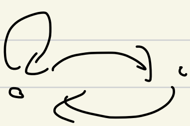
$=$



$A_G = [3]$,

capacity $\log_2 3$ bits

$=$



$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\log_2(\lambda_{PF}(A))$

is reasonable?

2 Tasks:

- if A is any irreducible matrix with non-negative entries:

  $\lambda_{pf}(A)$ is really an eigenvalue, > 0 if A not zero,

  $\#$ walks length n in A $\sim$ $\lambda_{pf}(A)^n \cdot c$

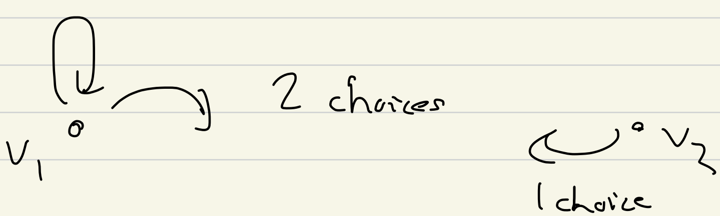
- apply this to convert binary data into constrained data

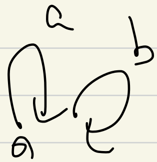
4-minute break

$$\frac{1+\sqrt{5}}{2} \approx 1.618033988\ldots$$

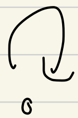
$$\frac{\log \frac{1+\sqrt{5}}{2}}{\log 2} = .69424\ldots \geq \frac{2}{3}$$

$$\text{capacity of Fib} = .69424\ldots \text{ bits}$$

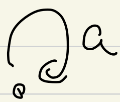




2 choices, capacity $\log_2 2 = 1$

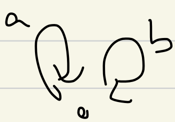


1 choice, capacity $\log_2 1 = 0$



word aaaaaaa

\# words length n = 1



aabbabbabbaab.. _

Fib graph $\longrightarrow$ $> \frac{2}{3}$ bits

3 (Fib graph) ~~a~~ little more 2 bits..

Data!

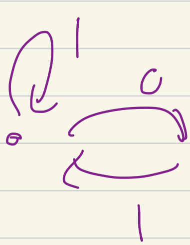
$\{0,1\}^{2m}$ 2m bits of arbitrary data

convert to

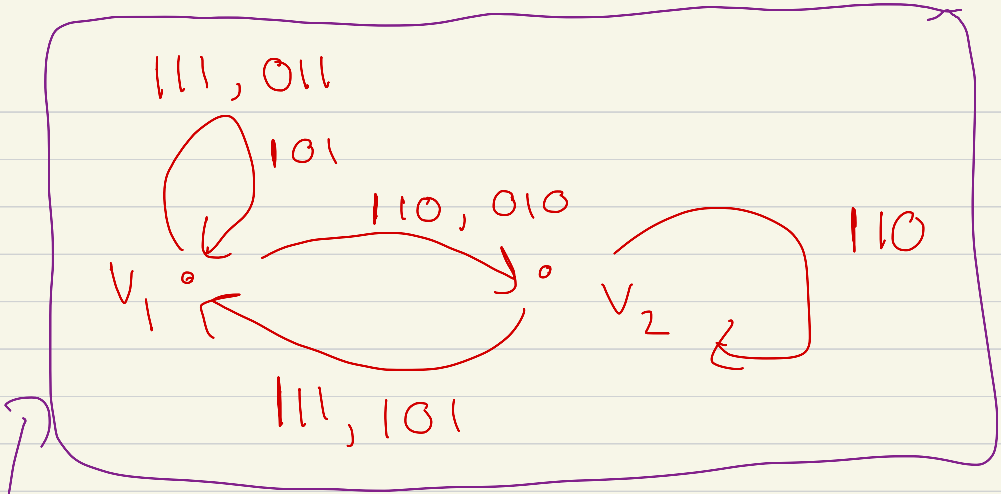
3m bits of Fib data

Since $\log_2(\lambda_{PF}) > \frac{2}{3}$

Walks of length 3 in Fib graph



walks of length 3



$A_{\text{walks of length 3 or Fib}} = \left[A_{fib}\right]^3$

0101001001100 1

x x x

x x x

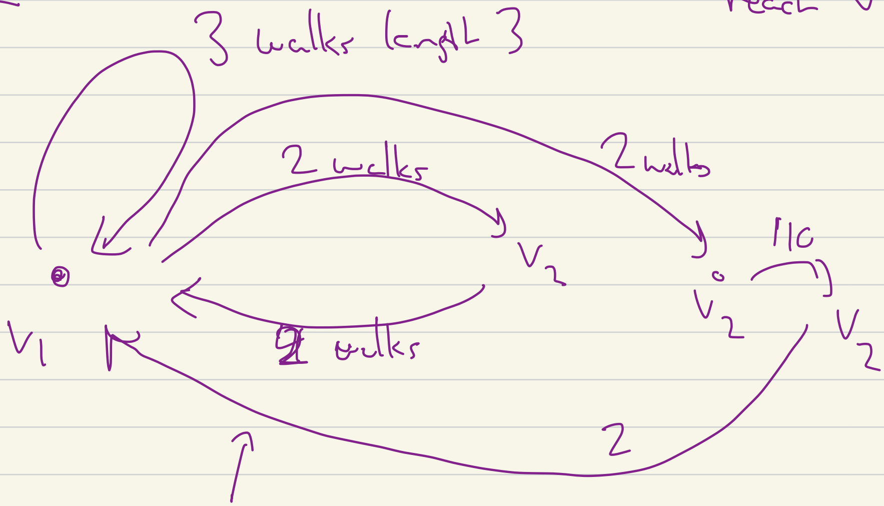
convert arbitrary data

fib data

ratio of 2 symbols

to 3 symbols

[start at v_1, end at v_1, stop first time reach v_1]



4 walks length 6   4 walks length 9

3 walks length 3

$$\begin{array}{ccc} 111 & \leftarrow & \overbrace{00}^{2} \\ 011 & \leftarrow & 01 \\ 101 & \leftarrow & 10 \end{array}$$

4 walks length 6

$$\begin{array}{ccc} 110\ 111 & \leftarrow & 11\ 00 \\ 010\ 111 & \leftarrow & 11\ 01 \\ 110\ 101 & \leftarrow & 11\ 10 \\ 010\ 101 & \leftarrow & 11\ 11 \end{array}$$

$$\underbrace{01}\ \underbrace{10}\ \underbrace{10}\ \underbrace{11}\ \underbrace{10}\ \underbrace{11}\ \underbrace{00}\ \underbrace{11}\ \ldots$$

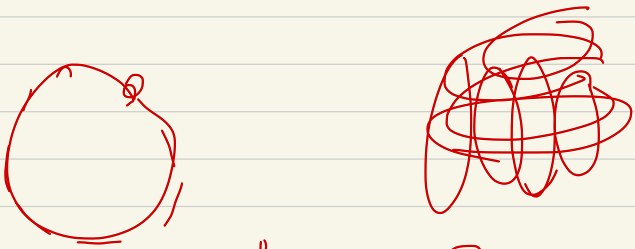
011   101   101   110101   110111

"sliding block" encoding

$$A_{Fib}^{3} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{matrix} \leftarrow 5 \text{ chars} \\ \leftarrow 3 \text{ chars} \end{matrix}$$

Class Ends



cycle length n coupling

$$\bar d(s+t) \le \bar d(s)\,\bar d(t)$$

couple   what is "uncouplable"

s steps p coupled
q uncoupled

t steps $\hat p$ coupled
$\hat q$ uncoupled

$s+t$ uncoupled $\Leftarrow$ $q\ \hat q$

$\mathbb{R}^n$: special eigenvectors:

$$\begin{pmatrix}1\\1\end{pmatrix} \text{ or } \begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} \text{ or } \begin{pmatrix}1\\-1\end{pmatrix} \otimes \cdots$$

$$\Updownarrow$$

$$\chi : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{C}^\times,$$

really $\{\pm 1\}$

$$\min \| A - u_1 v_1^T \|$$

$$A^T A v_1 = \lambda v_1 \qquad \lambda \geq 0$$

$$A A^T u_1 = \lambda u_1 \qquad ?$$

$$\operatorname{tr}\left((A - u_1 v_1^T)(A - u_1 v_1^T)^T \right)$$

$$= \operatorname{tr}(A A^T) - \lambda$$

$$A - u_1 v_1^T - u_2 v_2^T$$

$$A_1 := A - u_1 v_1^T$$

$$A_1 v_1 = (A - u_1 v_1^T) v_1$$

$$= A v_1 - u_1 (v_1 \cdot v_1)$$

$$= \underbrace{}_{u_1 (v_1 \cdot v_1)} \qquad = 0$$

$$A_1 \vec{v}_1 = 0$$

$$\vec{u}_1^T A_1 = 0$$

$$A^T A \vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\underbrace{A_1^T A_1 \vec{v}_1}_{A_1^T \vec{0}} = 0 \cdot \vec{v}_1$$

$$\underbrace{A_1^T \vec{0}}_{0}$$

$$A_1^T A_1 \vec{v}_2$$

Symm $A^T A_1$ eigenvectors

$$0 \le \lambda_n \le \ldots \le \lambda_2 \le \lambda_1$$

$$A_1^T A_1$$

$$0 \le \lambda_n \le \ldots \le \lambda_3 \le \lambda_2$$

$$\lambda_1$$

$$A^TA = \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T$$

$$A_1^TA_1 = \left(\sum_{i=2} \;\; '' \;\; \cdot\cdot \;\; '' \right)$$

If $A_1 \neq 0$, $\lambda_2 > 0$

$\vec{v}_1, \ldots \vec{v}_n$ still ON eigenbasis

$A^TA \quad \lambda_1, \lambda_2, \ldots, \lambda_n$

$A_1^TA_1 \quad 0, \lambda_2, \ldots, \lambda_n$

u_2 picks off largest eigenvalue

A :

$\vec{v}_1$ maximizes R_A

$\vec{v}_2$ " R_{A_1} ,

$A_1 = A - \lambda_1 \vec{v}_1 \vec{v}_1^T$

$\lambda_2 = \lambda_3 = \_\_ \_\_$

$$A^T A = \begin{bmatrix} 5 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$A_1^T A_1 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$v_2 \in sp(\vec{e}_2, \vec{e}_3, \vec{e}_4)$

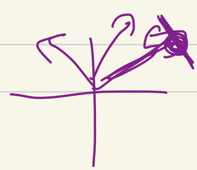
$$\left(\text{frob}\left(A_1 - \vec{u}_2 \vec{v}_2^{\,T}\right)\right)^2$$

$$\operatorname{Tr}\left(\left(A_1 - u_2 \vec{v}_2^{\,T}\right)\left(A_1^T - \vec{v}_2^{\,T} \vec{u}_2\right)\right)$$

$$R_{[1\ 2\ 3]} = \frac{x_1^2 + 2x_2^2 + 3x_3^2}{x_1^2 + x_2^2 + x_3^2}$$

$$\nabla(\quad) = \nabla(\quad) \quad \text{set } (0,0,0)$$

$\sim$



$$z_1^T + \varepsilon_1 \vec{u}_1 \qquad z_1^T + \varepsilon_2 \qquad \hat{A}$$

$$A - w_1 z_1^T - \cdots - w_{i-1} z_{i-1}^T - w_i (z_i + \varepsilon \vec{u})^T$$

$$- w_{i+1} z_{i+1}^T - \cdots$$

vary $z_i \rightsquigarrow (z_i + \varepsilon \vec{u})^T = z_i(\varepsilon)$

$\vec{u} \perp z_1, \ldots, z_{i-1}, z_{i+1}, \ldots$

$z_1, \ldots, z_{i-1}, z_i(\varepsilon), z_{i+1}, \ldots$ not orth

$$g(\varepsilon) = \left(\hat{A} - w_i (z_i + \varepsilon \vec{u})^T\right)\left(\hat{A} - w_i (z_i + \varepsilon \vec{u})^T\right)^T$$

the order ε term vanishes

$$\mathrm{Tr}\left(\left(\hat{A} - w_i (z_i)^T\right) \cdot u\, w_i^T\right)$$

$$+ \quad \text{same} \quad = 0$$

$$\mathrm{Tr}\left(\underbrace{\hat{A}\,\vec{u}\,\vec{w}_i^T}\ - w_i \underbrace{z_i^T u}_{z_i \cdot u} w_i^T \right.$$

$$\mathrm{Tr}(w_i w_i^T) =$$

$$\mathrm{Tr}(w_i^T w_i) = \vec{w}_i \cdot \vec{w}_i$$

$$\boxed{w_i^T \hat{A}\, \vec{u} = (\vec{z}_i \cdot \vec{u})(\vec{w}_i \cdot \vec{w}_i)}$$