April 1, 2021, CPSC 531F

- Today:
  - 3rd proof that symmetric matrices have ON eigenbasis, but more generally: $M \in \mathcal{M}_n(\mathbb{C})$, say that M is "normal" if

$$M^H M = M M^H$$

(symmetric, Hermitian, orthogonal, unitary, ...)

all have ON eigenbasis

"Schur decomposition"

- Perron-Frobenius thm:

If $A \in M_n(\mathbb{R})$ and has non-negative real entries, and all rows have at least one positive entry:

$\lambda_{\text{Perron-Frobenius}}$ eigenvalue,

if A is irreducible, then a bunch of nice things happen...

Lemma: (Schur decomposition)

Any $A \in M_n(\mathbb{C})$ can be written as

some stuff above the diagonal

$$AU = U \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ 0\text{'s} & & \ddots \ \lambda_n \end{bmatrix}$$

(where U is invertible)

$\lambda_1, \dots, \lambda_n$ are the eigenvalues of A

$$U = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix},$$

$\vec{v}_1, \dots, \vec{v}_n$ are ON as vectors in $\mathbb{C}^n$

So

$$\cancel{AU = UT} \qquad \left(\text{also } A = UTU^{-1} \right)$$

T is $\begin{bmatrix} \lambda_1 & & ??? \\ & \ddots & ??? \\ 0\text{'s} & & \lambda_n \end{bmatrix}$ is "upper triangular."

$$U^H U = \begin{bmatrix} - & \vec{v}_1^{\,H} & - \\ & \vdots & \\ - & \vec{v}_n^{\,H} & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0\text{'s} \\ & 1 & \\ & & \ddots \\ 0\text{'s} & & & 1 \end{bmatrix} = I$$

$$\vec{v}_i^{\,H} \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for $M_n(\mathbb{R})$ "orthogonal"

Say that $U \in M_n(\mathbb{C})$ is <u>unitary</u>

if (1) $U^H U = I$

(2) $U = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ with

$\vec{v}_1, \ldots, \vec{v}_n$ ON

(3) $U^H U = I = U U^H$

(4) $U = \begin{bmatrix} — & \vec{u}_1^H & — \\ & \vdots & \\ — & \vec{u}_n^H & — \end{bmatrix}$

$\vec{u}_1, \ldots, \vec{u}_n$ are ON basis

Proof (of Schur decomp):

$A \in M_n(\mathbb{C})$ has an eigenvalue λ,

corresponding eigenvector $\vec{v}$,

$A\vec{v} = \lambda\vec{v}$. Write $\lambda_1 = \lambda$, $\vec{v}_1 = \vec{v}/\|v_1\|$

[We know this char $\mathrm{poly}_A(\lambda) = \det(\lambda I - A)$,

--- $\Rightarrow \lambda$, $\exists\ \vec{v} \in \ker(\lambda I - A)$]

[Theory, not algorithm. That's practical...)

Extend $\vec{v}_1$, to an ON basis of $\mathbb{C}^n$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

Given $\vec{v}_1 = \begin{bmatrix} i/2 \\ -1/2 \\ i/\sqrt{2} \end{bmatrix}$,

(via Gram-Schmidt) ($n = 10^3, 10^6$ this may not be so practical)

$$A \underbrace{\begin{bmatrix} | & | & \\ \vec{v}_1 & \vec{v}_2 & \cdots \\ | & | & \end{bmatrix}}_{\text{unitary}} = \begin{bmatrix} | & | & \\ \vec{v}_1 & \vec{v}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \lambda_1 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? \end{bmatrix}$$

?

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$A\vec{v}_2$ = some combo of $\vec{v}_1, \ldots, \vec{v}_n$

$A\vec{v}_3 = \ldots$

Now: $A = A_1$

$$AU_1 = U_1 \left[\begin{array}{c|ccc} \lambda_1 & ? & \cdots & ? \\ \hline 0 & & & \\ 0 & & A_2 & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

$(n-1) \times (n-1)$ matrix

now use induction/recursion

A_2 : $\quad A_2 U_2 = U_2 \begin{bmatrix} \lambda_2 & ? & ? & ? \\ 0 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix}$

$$U_1^{-1} A U_1 = \left[\begin{array}{c|ccc} \lambda & ? & ? & ? \\ \hline 0 & & & \\ \vdots & & A_2 & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

$$\begin{bmatrix} & \\ & \end{bmatrix}^{-1} U_1^{-1} A_1 U_1 \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & U_2 \\ 0 & \end{bmatrix} = \begin{bmatrix} \lambda_1 & ???? \\ 0 & \lambda_2 \; ??? \\ \vdots & 0 \quad \\ 0 & 0 \quad A_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & U_2^{-1} \end{bmatrix}$$

keep going.

$=$

$n = 10^8$

$$U_1^{-1} A U_1 = \begin{bmatrix} \lambda_1 & ??? \\ 0 & \\ 0 & A_2 \\ 0 & \end{bmatrix}$$

$$A_2 = \begin{bmatrix} ??? \\ ??? \end{bmatrix} \cap \quad A_2 U_2 = U_2 \begin{bmatrix} \lambda_2 & ??? \\ 0 & \\ 0 & A_3 \\ 0 & \end{bmatrix}$$

$$\cdots \begin{bmatrix} 1 & & \\ & 1 & \\ & & u_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & \\ & u_2^{-1} \end{bmatrix} A \overbrace{u_1 \begin{bmatrix} 1 & \\ & u_2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & u_3 \end{bmatrix}}^{U} \cdots$$

gives you

$$\begin{bmatrix} \lambda_1 & ? & ? & ? & ? \\ & \ddots & & & ? \\ & & \ddots & & ? \\ & 0's & & & \lambda_n \end{bmatrix} = T$$

$$\longrightarrow U^{-1} A U = T$$

Algorithmically this is not so good ...

If $\quad A\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} T$

$$U^{-1}AU = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \\ 0 & t_{22} & t_{23} & \\ 0 & 0 & t_{33} & \\ \vdots & \vdots & \vdots & t_{nn} \end{bmatrix} = T$$

then if

$\boxed{A^H A = A A^H}$, i.e. A is normal

then

$$T^H T = (U^{-1}AU)^H (U^{-1}AU)$$

$(U^{-1} = U^H)$ $\qquad (U^H A U)^H (U^{-1}AU)$

$$= U^{-1} A^H A U$$

$T^H T = T T^H$ then T is also normal.

must be 0

all 0

$$\begin{bmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \cancel{\bar{t}_{12}} & \bar{t}_{22} & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \\ \end{bmatrix} \begin{bmatrix} t_{11} & \cancel{t_{12}} & \cdots & \\ 0 & t_{21} & \cdots & \\ 0 & 0 & & \\ \vdots & \vdots & & \end{bmatrix}$$

$$= \begin{bmatrix} |t_{11}|^2 & & \\ & |t_{12}|^2 + |t_{22}|^2 & \\ & & \ddots \end{bmatrix}$$

Similarly $(2,2)$ entry of $T^H T = T T^H$

$$TT^H$$

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & t_{23} & \cdots \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \bar{t}_{11} & 0 \\ \bar{t}_{12} & \bar{t}_{22} \\ \vdots & \bar{t}_{23} \\ \vdots & \vdots \end{bmatrix}$$

1st diagonal

$$= \begin{bmatrix} |t_{11}|^2 + |t_{12}|^2 + \cdots & \\ & 2,2, \text{ entry } \Rightarrow \\ & \end{bmatrix} \quad t_{23} = 0 = t_{24} = \cdots$$

Look at 1,1 entry of $T^H T = T T^H$

$$|t_{11}|^2 = |t_{11}|^2 + \underbrace{|t_{12}|^2 + \cdots}_{\text{all } 0} \Rightarrow$$

So!

T is upper tri, and normal

(1) $T^H T = T T^H \Rightarrow T$ is diagonal

(2) If $AU = UT$, U unitary,

$A^H A = A A^H \Rightarrow T$ is normal

"Schur decomposition"

$\Rightarrow$ any normal A has ON eigenbasis e.g. $\underbrace{U^{-1} = U^H}$, $\quad U^H U = I$

then

$$U^H U = I = U U^H$$ normal

So U is normal.

$=$

A is skew-Hermitian if

$$A^H = -A$$ normal

$$A^H A = A A^H = -A^2$$

Perron-Frobenius:

Let $A \in M_n(\mathbb{R})$ with non-neg entries

e.g. A Markov matrix

A adj mat (di)graph

A from $(2,7)$-codes
information

We define for such A:

$\lambda_{PF}(A)$:

(1) λ_{pf} is the largest real eigenvalue of A

Then we assume A is irreducible, and prove

- any eigenvalue of A has absolute value at most λ_{pf}

- λ_{pf} has multiplicity 1

- λ_{pf} has unique eigenvector, $\vec{v}$, up to scaling, and we can take $\vec{v} > \vec{0}$, $\vec{v}$ to

have all real, positive components...

Let's define, for any $A \in M_n(\mathbb{R})$

with non-negative entries!

$$PF(A) = \left\{ \begin{array}{c} \lambda \in \mathbb{R} \\ \lambda \geq 0 \end{array} \;\middle|\; \begin{array}{c} A\vec{v} \geq \lambda \vec{v} \text{ for} \\ \text{some } \vec{v} \geq 0 \end{array} \right\}$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\quad A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \geq 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so $3 \in PF(A)$

We look at

$$\max \lambda \in PF(A)$$

and use λ_{PF} for this maximum value,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \geq \frac{4}{5} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \geq \left(\frac{3}{7}\right) \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \geq 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad \lambda = a+b,\ a-b$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 13 \\ 23 \end{bmatrix}$$

$$= \text{something} \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$\frac{23}{7}$ not quite 4

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = (n) \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

close to 4

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} = 4 \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 20 \\ 20 \end{bmatrix} = 4 \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

λ_{pf} eigenvalue!

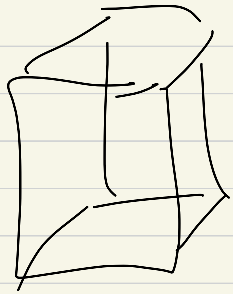
$$A \begin{bmatrix} \text{non-neg entries} \\ \\ \text{"} \qquad \text{"} \end{bmatrix} \geq \begin{pmatrix} \text{as} \\ \text{large} \end{pmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

as possible

Class ends

G Cayley graph,

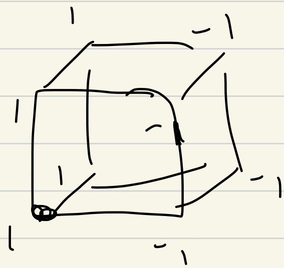
$G = \mathbb{B}^n$



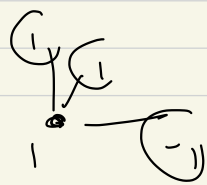
$\mathbb{B}^3$

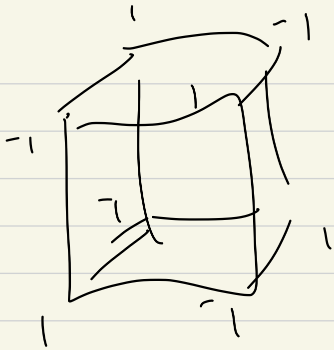
Graph $(\mathbb{Z}/2\mathbb{Z})^3 \leftarrow$ group

A_G eigenvectors

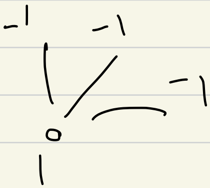


$\lambda = 1$





$\lambda = -3$



Eigenvectors with ± 1 values

$$(\mathbb{Z}/2\mathbb{Z})^3 \xrightarrow{\chi} \mathbb{C}$$

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$$

G based on Abelian group,

$f : V \to \mathbb{R}$

f = sum $\cancel{of}$ $\sum c_i \vec{v}_i$

eigenvectors

f ( ) if $\vec{v}_1, \ldots, \vec{v}_n$ ON

$$f = \sum_{i=1}^{n} \underbrace{(f \cdot \vec{v}_i)}_{} \vec{v}_i$$

→ "fourier coefs of f"

$$\hat{f}(\chi) := (f \cdot \chi) \text{ maybe } / \sqrt{n}$$

G is Abelian, then a <u>character</u>

of G is map

$$\chi : G \to \mathbb{C}^{\text{non-zero}}$$

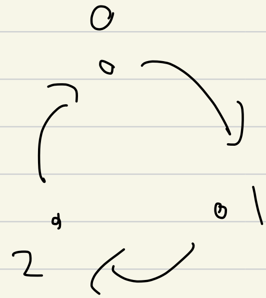
$$\boxed{\chi(g_1 g_2) = \chi(g_1)\,\chi(g_2)}$$

$G = \mathbb{R}$:

$$\chi(g) = e^{(2\pi i \xi)\cdot g}$$

χ has values on unit circle

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

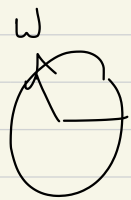


modulo 3,

$$g = \text{group} = \mathbb{Z}/3\mathbb{Z}$$

eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : \quad g \longmapsto 1^g$$

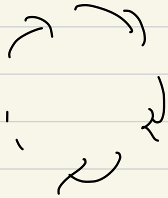


$$\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} : \quad g \longmapsto \omega^g$$

$$\begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \end{bmatrix} : \quad g \longmapsto (\omega^2)^g$$

$$\omega = e^{2\pi i/3}$$

C_n,



group $\mathbb{Z}/n\mathbb{Z}$

$\zeta^n = 1$

$$\chi_\zeta(g) \longmapsto \zeta^g$$

Fourier coefs

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

Fourier trans $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to$

$c_1 \leftarrow$ char $g \mapsto 1^g$

$c_2 \leftarrow g \mapsto \omega^g$

$c_3 \leftarrow g \mapsto \omega^{2g}$

Poss?

- 3 present, ~25 min last day April 13

-

One (or more) April 8

- Total Unimodularity

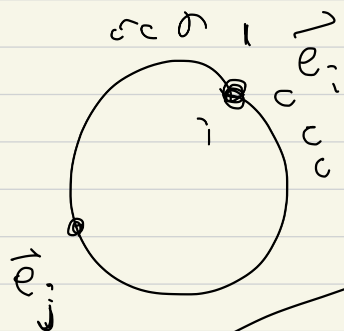
- Electrical Networks

Coupling!

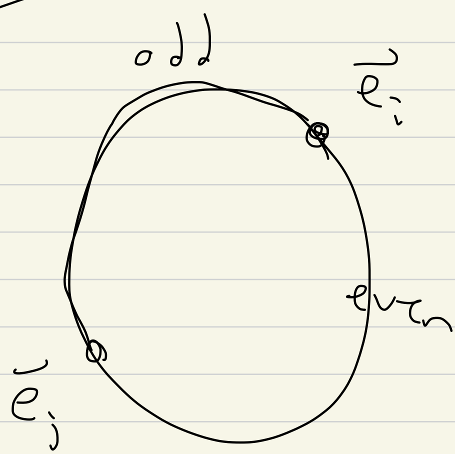
$$\bar{d}(s+t) \leq \bar{d}(s)\,\bar{d}(t) \qquad 4.11$$

~~Another coupling~~

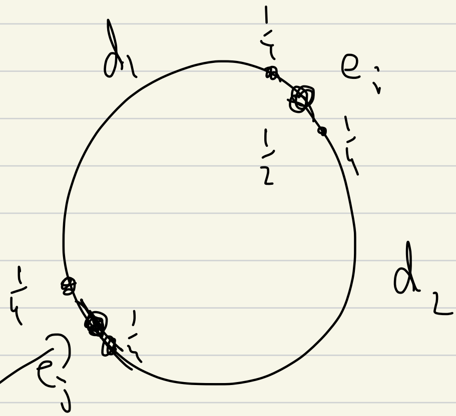
Coupling examples.



n odd



lazy $\frac{1}{2}$, left or right $\frac{1}{4}$



$d_1 \leq d_2$

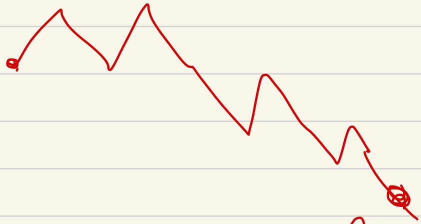
stays prob $\frac{1}{2}$

n stop

$k \longrightarrow$

0 —

stop



$$E[\text{time to stop}] = k(n-k)$$

$$\text{Prob}[\text{time to stop} \geq t] \cdot t \leq E[\text{time to stop}]$$

X takes on non-neg values

$$E[X] = 0 \cdot \text{Prob}(X=0)$$
$$+ 1 \cdot \text{Prob}(X=1)$$
$$+ \vdots$$

any t $\geq t \cdot \text{Prob}(X=t)$
$$(t+1) \cdot \text{Prob}(X=t+1)$$
$$+ \vdots$$

$$\geq t \ \text{Prob}(X=t)$$
$$+ t \ \text{Prob}(X=t+1)$$
$$+ \vdots$$

$$= t \cdot \text{Prob}(X \geq t)$$

$$\text{Prob}\left(\begin{matrix}\text{time to} \\ \text{stop}\end{matrix} \geq t\right) \leq \frac{k(n-k)}{t}$$

Gambler's ruin!

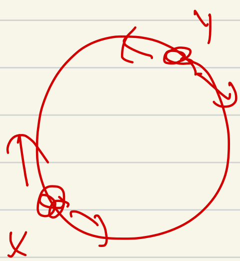
$$\text{Prob}(\text{time to stop} \geq t) \leq \frac{k(n-k)}{t} \quad \text{max all } k$$

$$\leq \frac{(n/2)(n - \frac{n}{2})}{t}$$

$$\lesssim \frac{n^2}{4t}$$

Gambler's ruin formula

Mixing time

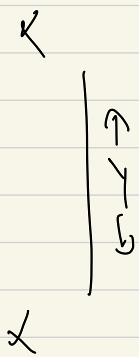


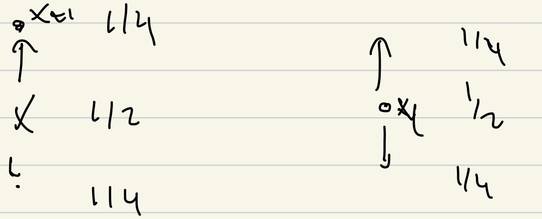
lazy!

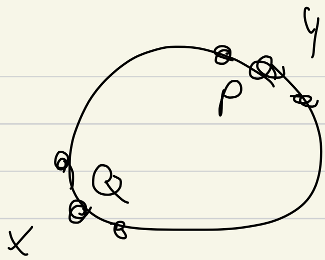
if $x = y$, either $\begin{pmatrix} x \text{ moves} \\ y \text{ stays} \end{pmatrix}$ or $\begin{pmatrix} x \text{ lazy} \\ y \text{ moves} \end{pmatrix}$

Time for $Y = X \leq \dfrac{n^2}{4t}$

$$\max_{i,j} \left| \vec{e}_i P^t - \vec{e}_j P^t \right| \leq$$ time when gamblers ruin has prob $1/4$







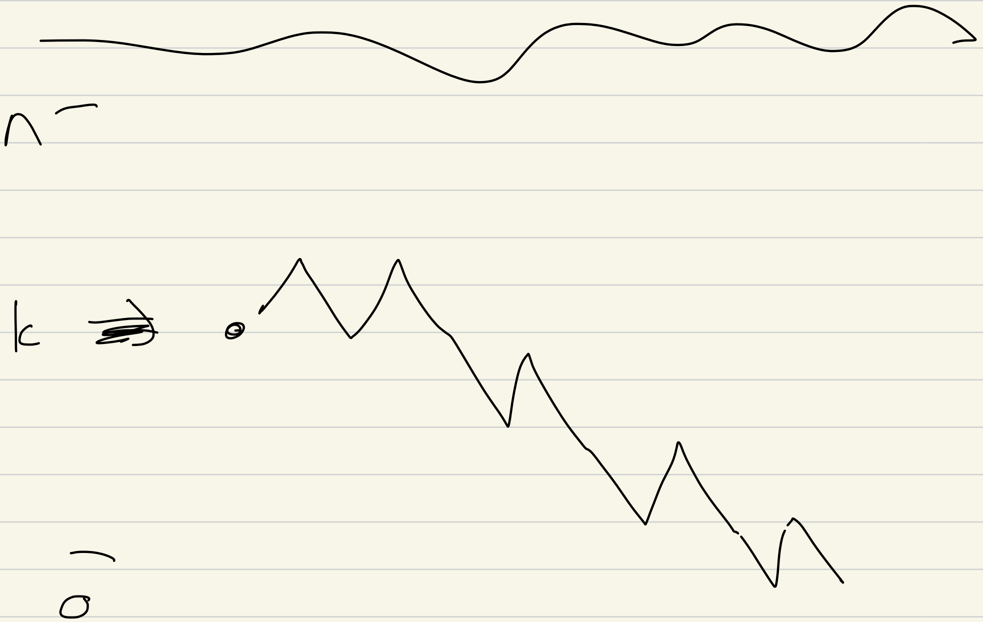
$$Y_{t+1} = P-1,\ P,\ P+1$$

$$X_{t+1} = Q-1,\ Q,\ Q+1$$

$$\left.\begin{aligned} \text{Prob}(P, Q) &= \tfrac{1}{4} \\ \text{Prob}(P, Q+1) &= 1/8 \\ \text{Prob}(P+1, Q+1) &= 1/16 \end{aligned}\right\} \text{ If } X, Y \text{ independent}$$

Mixing time: $\max |\vec{e}_i P^t - \vec{e}_j P^t| \le \frac{1}{4}$

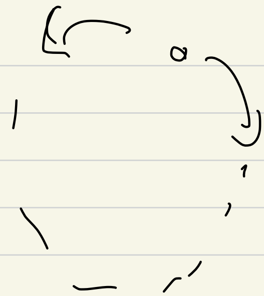
if f is the Prob $\leq \frac{1}{4}$ point for Gambler's ruin.



f time to reach 0 or n

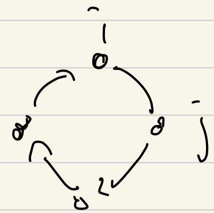
is $k(n-k)$

Mixing time on $\rho = (C_n + C_n^{-1})\frac{1}{2}$

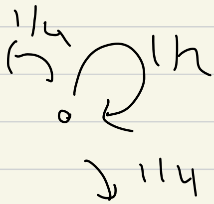


n even

mixing time = ∞



now $\quad P = \frac{1}{4}\left(C_n + C_n^{-1} + 2I \right)$



$$\max_{i,j} \left| \vec{e}_i P^t - \vec{e}_j P^t \right| \le \frac{1}{4} \quad \text{when}$$

t is the $\frac{1}{4}$ point of gam mix

$\varepsilon = 0$ mixing time $= \infty$

Say $\frac{(1-\varepsilon)}{2} C_n + \frac{(1-\varepsilon)}{2} C_n^{-1} + \varepsilon I$

$P = \frac{2}{5} C_n + \frac{2}{5} C_n^{-1} + \frac{1}{5} I$

lazy prob $1/5$

mixing time?

X, Y!

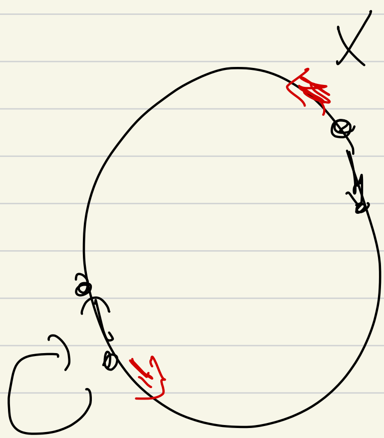
when X is lazy, pr $\frac{1}{5}$, Y moves

" Y " " pr $\frac{1}{5}$, X moves

$\frac{3}{5}$ @ X, Y move in opp dir

$\frac{2}{5}$ prob ! gamblers ruin move

$\frac{3}{5}$ prob : -- -- stay



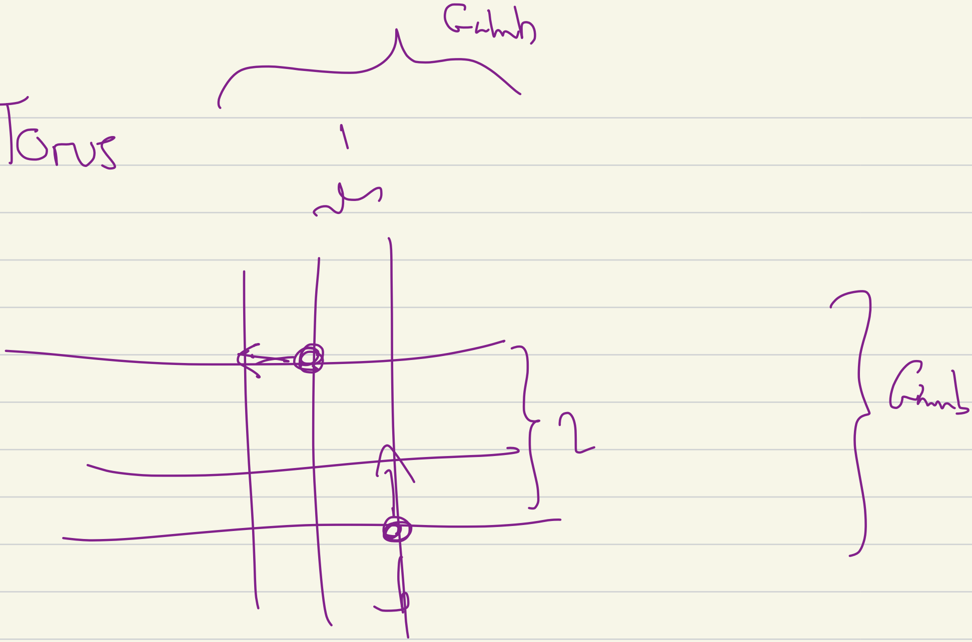
X lazy prob $\frac{1}{5}$ $\rightarrow$ $\frac{2}{5}$ $\rightarrow$ $\frac{1}{2}$

4 lazy $\frac{1}{5}$ $\rightarrow$ $\frac{3}{5}$ $\rightarrow$ $\frac{1}{2}$

after $t \simeq n$ steps of

binomial

Torus



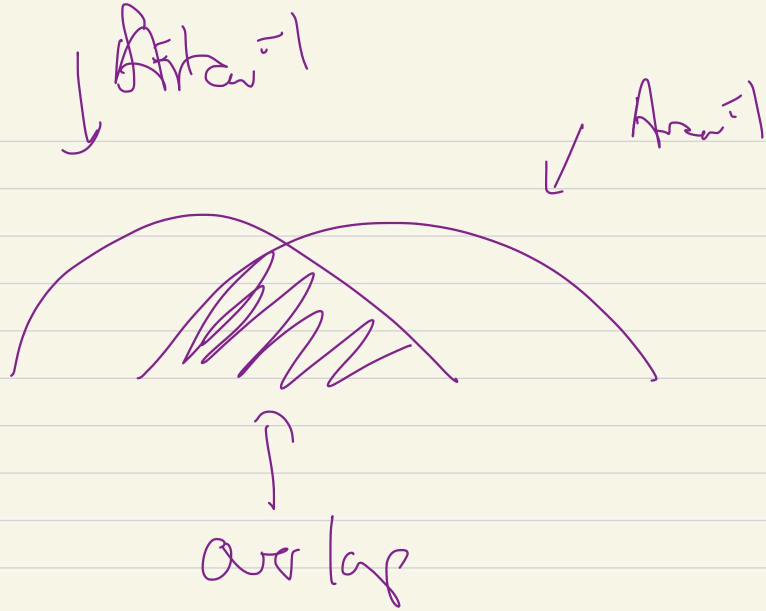
Coupling! X, Y

pick 1st coord or 2nd coord prob $\frac{1}{2}$ each

then

X is lazy prob $\frac{1}{2}$, Y moves

or — — —



"shower"

TV ⟷