CPSC 531F   March 30, 2021

Today:

- 2 other proofs that symmetric matrices are diagonalizable with an ON basis
- Perron-Frobenius thm

=

4 other classes
- Applications
- Student Presentations

some currently open research problems

Important idea!

Let's say that $A = A^T$, $A \in M_n(\mathbb{R})$

say that A has distinct eigenvalues,

$\lambda_1, \ldots, \lambda_n$.

① λ_i are real!

Define standard inner product on $\mathbb{C}^n$

$$\langle \vec{u}, \vec{v} \rangle = \vec{v}^{\,*} \vec{u} \quad \text{(Horne \& Johnson)}$$

$$= \vec{v}^{\,H} \vec{u} \quad (H = * = \text{conjugate transpose})$$

$$= u_1 \overline{v_1} + u_2 \overline{v_2} + \ldots + u_n \overline{v_n}$$

$$\overline{a+ib} = a - ib, \quad a, b \in \mathbb{R},$$

Warning: $\langle \alpha \vec{u}, \vec{v} \rangle$, $\alpha \in \mathbb{C}$

$$= \alpha \langle \vec{u}, \vec{v} \rangle$$

but

$$\langle \vec{u}, \alpha \vec{v} \rangle = \overline{\alpha} \langle \vec{u}, \vec{v} \rangle$$

BUT, THERE IS NO UNIFORM CONVENTION ON THIS! ELSEWHERE

$$\langle \vec{u}, \vec{v} \rangle := \overline{u}_1 v_1 + \dots + \overline{u}_n v_n$$

then $\langle \alpha \vec{u}, \vec{v} \rangle = \overline{\alpha} \langle \vec{u}, \vec{v} \rangle \dots$

We want!

$$\langle \vec{u}, \vec{u} \rangle = |u_1|^2 + \dots + |u_n|^2$$

So

$$u_1 \overline{u_1} + u_2 \overline{u_2} + \dots$$

$$=$$

$$\overline{u_1} u_1 + \overline{u_2} u_2 + \dots$$

If $A = A^T$, $A \in M_n(\mathbb{R})$, and

$A\vec{u} = \lambda \vec{u}$, $\vec{u} \neq 0$ but $\lambda \in \mathbb{C}$

(since λs of rotation generally complex

--- large Markov chain)

$$\langle A\vec{u}, \vec{u}\rangle = \vec{u}^H A\vec{u}$$

$$= (A^H\vec{u})^H \vec{u}$$

$$= \langle \vec{u}, A^H\vec{u}\rangle$$

$$= \langle \vec{u}, A\vec{u}\rangle$$

Remark!

$$\langle A\vec{u}, \vec{u}\rangle = \langle \vec{u}, A\vec{u}\rangle$$

only really use $\boxed{A^H = A}$ ← Hermitian

$$(\bar{A})^T$$

If $A \in M_n(\mathbb{R})$, $A^T = A$

or $A \in M_n(\mathbb{C})$, $A^H = A$

then

$$\boxed{\langle A\vec{u}, \vec{u} \rangle = \langle \vec{u}, A\vec{u} \rangle}$$

also

$$\boxed{\langle A\vec{u}, \vec{v} \rangle = \langle \vec{u}, A\vec{v} \rangle}$$

e.g.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 1+7i \\ 1-7i & 2 \end{bmatrix}$$

$A^T = A$, real, $\quad A^H = A$

1. $A\vec{v} = \lambda\vec{v}, \quad \vec{v} \neq 0, \quad \lambda \in \mathbb{R}$

if A real symmetric (or Hermitian)

$$\langle A\vec{v}, \vec{v}\rangle = \langle \vec{v}, A\vec{v}\rangle$$

$$\parallel$$

$$\langle \lambda\vec{v}, \vec{v}\rangle = \langle \vec{v}, \lambda\vec{v}\rangle$$

$$\parallel$$

$$\lambda \underbrace{\langle \vec{v}, \vec{v}\rangle}_{\neq 0} = \bar{\lambda} \underbrace{\langle \vec{v}, \vec{v}\rangle}_{\neq 0}$$

$\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$

② If $A\vec{v} = \lambda \vec{v}$

and $A\vec{w} = \mu \vec{w}$

and $\lambda \neq \mu$, then $\vec{v} \perp \vec{w}$:

$$\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A\vec{w} \rangle$$

$$\parallel$$

$$\langle \lambda \vec{v}, \vec{w} \rangle = \langle \vec{v}, \mu \vec{w} \rangle$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\lambda \langle \vec{v}, \vec{w} \rangle \qquad\qquad \langle \vec{v}, \vec{w} \rangle \bar{\mu}$$

$$\parallel$$

$$\langle \vec{v}, \vec{w} \rangle \mu$$

$$(\lambda - \mu) \langle \vec{v}, \vec{w} \rangle = 0$$

$$\lambda \neq \mu \Rightarrow \langle \vec{v}, \vec{w} \rangle = 0$$

So A has distinct eigenvalues

$$\lambda_n < \cdots < \lambda_2 < \lambda_1$$

A has eigenvectors $A\vec{v}_i = \lambda_i \vec{v}_i$

and $\vec{v}_i \perp \vec{v}_j$ for $i \neq j$

So $\vec{v}_1, \ldots, \vec{v}_n$ are mutually orthogonal

Could take $\hat{v}_i = \vec{v}_i / |\vec{v}_i|$

We've proved: $A = A^T$ real

or

$A = A^H$

then

if A has distinct eigenvalues,

then A has ON eigenbasis.

Principle: If A is $A = A^T$ real

or $A = A^H$, there exist

$A_1, A_2, \ldots$

s.t. (1) $\lim_{m \to \infty} A_m = A$

② A_m are $\begin{Bmatrix} \text{Hermitian} \\ \text{real symmetric} \end{Bmatrix}$

if A is

Idea

$$A(\varepsilon) = A + \overbrace{\begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & \ddots & \\ & & & & n \end{bmatrix}}^{\text{real symmetric} \quad \text{Hermitian}} \varepsilon$$

consider

$$A_1 = A(1), \quad A_2 = A\left(\tfrac{1}{2}\right), \quad A_3 = A\left(\tfrac{1}{3}\right),$$

$\cdots$. So $A_m = A\left(\frac{1}{m}\right) \to A$ $\quad m \to \infty$

Claim: $A(\varepsilon)$, $\varepsilon \in \mathbb{R}$ has distinct eigenvalues for all but at most finitely values of ε :

Moreover : $B \in M_n(\mathbb{R}), M_n(\mathbb{C})$,

B has distinct eigenvalues

$\iff \text{poly}_n(b_{11}, b_{12}, \ldots, b_{1n}, b_{21}, \ldots, b_{nn})$

$= 0$

$\neq$

$\iff$ ~~char~~ $\text{poly}_B(\lambda)$ has

distinct roots

$p(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \ldots + c_n\lambda^n$,

$c_n \neq 0$, p has distinct roots iff

$p'(\lambda)$ and $p(\lambda)$ don't have a common root

e.g.

$$A\lambda^2 + B\lambda + C = 0$$

double-root iff $\quad B^2 - 4AC = 0$

In particular

$$A(\varepsilon) = A(1-\varepsilon) + \varepsilon \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & \ddots \\ & & & & n \end{bmatrix}$$

then $A(\varepsilon)$ has distinct roots

iff some poly in entries of

$$\text{poly}\begin{pmatrix} (1-\varepsilon)a_{11}+\varepsilon & (1-\varepsilon)a_{12} & (1-\varepsilon)a_{13} & \cdots & (1-\varepsilon)a_{1n} \\ (1-\varepsilon)a_{21} & (1-\varepsilon)a_{22}+2\varepsilon & \cdots & & \\ \vdots & & \ddots & & \\ & & & & (1-\varepsilon)a_{nn}+\varepsilon n \end{pmatrix}$$

is 0. So distinct eigenvalues

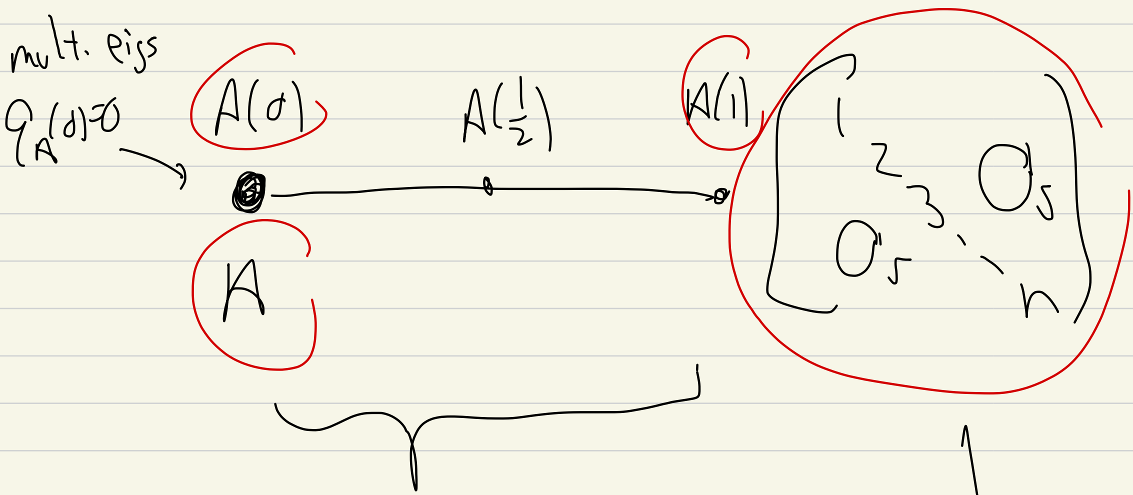
iff poly $q_A(\varepsilon)=0$.

But $A(1) = \cancel{A(1-1)} + 1\cdot\begin{bmatrix}1 & & \\ & 2 & \\ & & \ddots & \\ & & & n\end{bmatrix}$

this has distinct roots. So $q_A(\varepsilon)$

is not the zero polynomial.

Hence $q_A(\varepsilon)$ has only finitely many roots.



"homotopy" $\quad A(\varepsilon) = A\cdot(1-\varepsilon) + (\quad)\varepsilon$

$$A(0) = A$$

$$A(1) = \begin{bmatrix} 1 & & & \\ & 2 & & 0's \\ & & \ddots & \\ 0's & & & n \end{bmatrix}$$

$$\boxed{\text{① } A = A^H \quad (\text{or } A = A^T \text{ real}) \\ \text{if } A \text{ has distinct eigenvalues,} \\ \exists \text{ ON eigenbasis for } A}$$

② There is $A_1, A_2, \dots A_m \longrightarrow A$ Perturbation

$$\begin{array}{ccc} A_1 & A_2 & A_m \\ \downarrow & \downarrow & \downarrow \\ A(1) & A(\tfrac{1}{2}) & A(\tfrac{1}{m}) \end{array}$$

throw out any $A(\frac{1}{m})$ with multiple eigenvalues.

So each A_m has ON eigenbasis.

③ If you have

A_m have ON eigenbasis

$$\vec{V}_{m,1}, \vec{V}_{m,2}, \ldots, \vec{V}_{m,n}$$

then a subsequence of

$$m_1, m_2, m_3 \to \infty$$

$$\vec{V}_{m_j, i} \longrightarrow \vec{V}_i$$

and

$\vec{V}_1, \ldots, \vec{V}_n$ is an ON eigenbasis

for A.

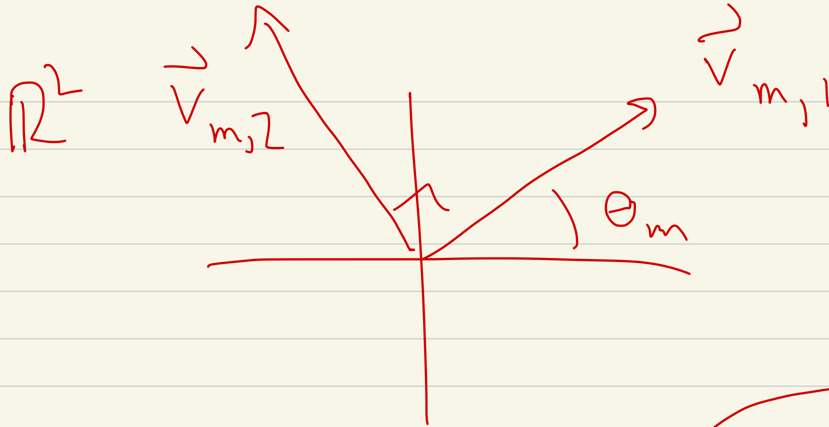
Rem:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$A(\varepsilon) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}(1-\varepsilon) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\varepsilon$$

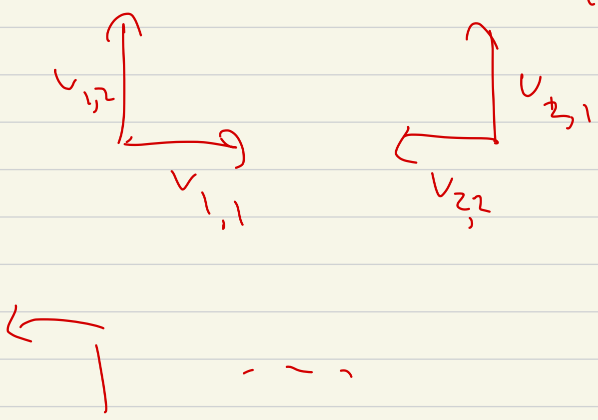
$$= \begin{bmatrix} 3-\varepsilon & 0 \\ 0 & 3-2\varepsilon \end{bmatrix}$$

get each $A(\varepsilon)$, ~~exc~~ except $3-\varepsilon = 3-2\varepsilon$ $(\varepsilon=0)$ has distinct eigenvalues, eigenvectors $\vec{e}_1, \vec{e}_2$



$$\Theta_m = m \pmod{2\pi}$$

$$\frac{2\pi}{4} m \pmod{2\pi}$$



not necessarily limit to a sequence of ON vectors, but there is a subsequence that converges ⇒ COMPACTNESS

of set of ON-eigenvectors

i.e.

$O(n)$ = group $n \times n$ orthogonal matrices,

this is <u>compact</u>

4 minute break

— Perron-Frobenius thm

- 3rd proof involves

"normal matrices"

When does $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ test

have multiple roots?

$$\mathrm{char}_A(\lambda) = \det(\lambda I - A)$$

$$= \det \begin{bmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{bmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22}) - a_{21}a_{12}$$

$$= \lambda^2 \underset{\substack{\downarrow \\ 1}}{c_2} + \lambda \underset{\substack{\downarrow \\ -\mathrm{trace}(A)}}{c_1} + \underset{\substack{\searrow \\ \det(A)}}{c_0}$$

$= p_A(\lambda)$.

When does this have distinct roots

$$\boxed{\underbrace{(-\mathrm{trace}(A))^2 - 4 \cdot 1 \cdot \det(A)}_{P(a_{11}, a_{12}, a_{21}, a_{22})} \neq 0}$$

P can't be the zero poly:

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has roots in its char poly

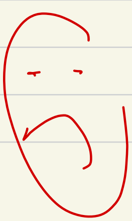
$$A(\varepsilon) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} (1-\varepsilon) + \varepsilon \begin{bmatrix} 7 & 0 \\ 0 & 8 \end{bmatrix}$$

$A(\varepsilon)$ has multiple eigenvalues

four entries

iff $p(\quad , \quad , \quad , \quad) = 0$

$$A(\varepsilon) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \varepsilon \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$



$$= \begin{bmatrix} 4+10\varepsilon & 0 \\ 0 & 4+10\varepsilon \end{bmatrix}$$

always has multiple eigenvalues

$$\left(-\operatorname{trace}(A)\right)^2 - 4\cdot 1\cdot \det(A) = 0$$

$A(\varepsilon)$ $\qquad\qquad$ $A(\varepsilon)$

here

$$\left((4+10\varepsilon)\cdot 2\right)^2 - 4\cdot 1\cdot (4+10\varepsilon)^2$$

$$= 0$$

$$A(\varepsilon) = \text{anything}\,(1-\varepsilon) + \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\varepsilon$$

four entries

$$\mathbb{P}(\ \ ,\ \ ,\ \ ,\ \ ) = p(\varepsilon)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \leftarrow \text{☺}$$

anything

0 ———— ε

?

$\varepsilon = 1$, $A(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ distinct eigs

any

$$A(\varepsilon) = \begin{bmatrix} \text{poly}(\varepsilon) & \widehat{\text{poly}}(\varepsilon) & \cdots \\ \vdots & & \\ & & \end{bmatrix}$$

More generally

A 3×3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

there is

$$P(\text{ 9 entries }) = 0$$

iff A has at least

one multiple eigenvalue

Class ends