March 11:

- Today start 3 proofs that symmetric matrices, $M_n(\mathbb{R})$, have ON eigenbases

1st Proof Rayleigh quotients.

Recall! $\vec{v} \in \mathbb{R}^n$,

norms $$\|\vec{v}\|_p := \left(|v_1|^p + \dots + |v_n|^p\right)^{1/p}$$

norm!

weight $\vec{\omega} \in \mathbb{R}^n$, $\vec{\omega} = (\omega_1, \dots, \omega_n)$

all $\omega_i > 0$

$$\|\vec{v}\|_{\vec{\omega}} := \left(|v_1|^2\omega_1 + \dots + |v_n|^2\omega_n\right)^{1/2}$$

comes from

$$(\vec{x}, \vec{y})_{\vec{w}} := x_1 y_1 w_1 + \dots + x_n y_n w_n$$

" $\vec{w}$ weighted dot product "

$$(\vec{x}, \vec{y})_{\vec{1}} := x_1 y_1 + \dots + x_n y_n$$

$=$

Last time: linear map $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$

(i.e. $\vec{v} \mapsto A\vec{v}$ some A)

$$R_{\mathcal{L}}(\vec{v}) = \frac{(\mathcal{L}\vec{v}, \vec{v})}{(\vec{v}, \vec{v})}$$

for any inner product $(\, , \,)$: for us, mostly interested in

$(\vec{x}, \vec{y})_{\vec{1}}$ = standard inner product $= \vec{X} \cdot \vec{Y}$,

but $P \in M_n(\mathbb{R})$ is ~~Markov~~ Markov, irreducible,

there is a unique stationary distribution

$\vec{\pi}$ of P, we are interested in

$$(\ , \ )_{\vec{\pi}} \quad \text{and} \quad (\ , \ )_{1/\vec{\pi}}$$

reason is:

$$(P\vec{x}, \vec{y})_{\vec{\pi}} = (\vec{x}, P\vec{y})_{\vec{\pi}}$$

and

$$(\vec{x}^T P, \vec{y}^T)_{1/\vec{\pi}} = (\vec{x}^T, \vec{y}^T P)_{1/\vec{\pi}}$$

Thm! If inner product, $(\, , \,)$, on $\mathbb{R}^n$

and $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation

that is self-adjoint, i.e.

$$(\mathcal{L}\vec{x}, \vec{y}) = (\vec{x}, \mathcal{L}\vec{y}) \quad \text{for all } \vec{x}, \vec{y}$$

then $\mathcal{L}$ has an ON-eigenbasis

ON means wrt $(\, , \,)$: $\vec{v}_1, \ldots, \vec{v}_n$ ON if

$$(\vec{v}_i, \vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Think of $(\, , \,)$ on any vector space,

V, ($\mathbb{R}$-vector space or $\mathbb{C}$-vector space)

and $\dim(V)$ finite.

HW:

$$(\sigma - 1)\binom{n}{3}$$

$$= \binom{n+1}{3} - \binom{n}{3} =$$

$$= \frac{(n+1)n(n-1) - n(n-1)(n-2)}{3!}$$

$$= \frac{n(n-1)\cdot 3}{3!} = \frac{n(n-1)}{2!} = \binom{n}{2}$$

$$\left((\sigma-1)\binom{n}{k} = \binom{n}{k-1} \right)$$

$$(\sigma-1)\left[(\sigma-1)\binom{n}{3}\right] = (\sigma-1)\binom{n}{2} = \binom{n}{1}$$

$$(\sigma-1)^3\binom{n}{3} = (\sigma-1)\binom{n}{1} = \binom{n}{0} = 1$$

$$(\sigma-1)^4\binom{n}{4} = 0$$

$$\underbrace{(\sigma-1)^3\binom{n}{3}} = 0 \quad - - -$$

$$\sigma^3\binom{n}{3} = 3\sigma^2\binom{n}{3} - 3\sigma\binom{n}{3} + \binom{n}{3}$$

etc.

Last time claim:

$$\mathcal{R}_{\mathcal{L}}(\vec{v}) := \frac{(\mathcal{L}\vec{v}, \vec{v})}{(\vec{v}, \vec{v})}$$

then if $\alpha \in \mathbb{R}$, $\alpha \neq 0$,

$$\mathcal{R}_{\mathcal{L}}(\alpha\vec{v}) = \frac{(\mathcal{L}(\alpha\vec{v}), \alpha\vec{v})}{(\alpha\vec{v}, \alpha\vec{v})}$$

$$= \frac{\alpha^2 \, (\mathcal{L}\vec{v}, \vec{v})}{\alpha^2 \, (\vec{v}, \vec{v})} = \mathcal{R}_{\mathcal{L}}(\alpha\vec{v})$$

Think of $\mathcal{L}: \vec{v} \to \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \vec{v}$,

$$\mathcal{R}_{\mathcal{L}}(\vec{v}) = \frac{((d_1 v_1, d_2 v_2, \ldots), (v_1, \ldots, v_n))}{((v_1, \ldots, v_n), (v_1, \ldots, v_n))}$$

[for standard inner prod!

$$\frac{d_1 v_1^2 + d_2 v_2^2 + \_\_ + d_n v_n^2}{v_1^2 + \_\_ + v_n^2}$$
]

Claim 1: If maximum value of

$R = R_{\mathcal{L}} : \mathbb{R}^n \to \mathbb{R}$ is attained

at $\vec{v}$, then

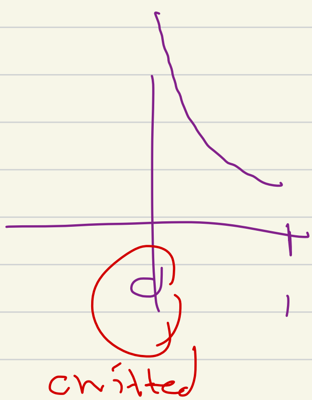
(1) $\mathcal{L}\vec{v} = \lambda \vec{v}$

(2) $\lambda = R_{\mathcal{L}}(\vec{v})$.

Remark: If f is continuous function from $[a,b] \subset \mathbb{R}$ to $\mathbb{R}$, then f attains its maximum value somewhere.

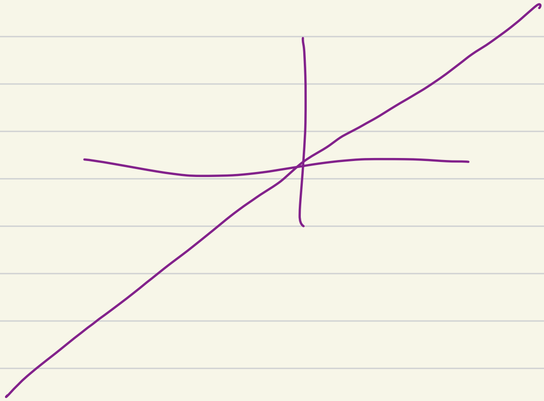
Rem: $f(x) = \frac{1}{x}$ on $(0,1)$

$(0,1) = \{ x \mid 0 < x < 1 \}$



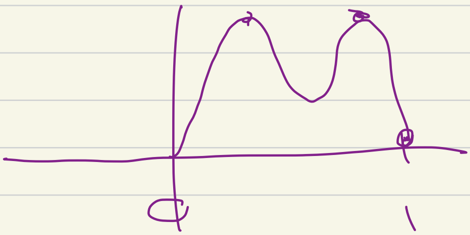
has no maximum value

Rem! $f(x) = x$, $\mathbb{R} \to \mathbb{R}$



no maximum value

f:



$f(x) = 0.6$

Idea: If you have any sequence

$x_1, x_2, \ldots$ all in $[a,b]$

think of $[0,1]$

there is a subsequence

$x_{i_1}, x_{i_2}, \ldots$ that converges

$i_1 < i_2 < \ldots$

I.e. $[a,b]$ is <u>sequentially</u> <u>compact</u>.

Then: (1) f continuous on $[a,b]$, $\Rightarrow$ f is bounded

(2) any bounded subset in $\mathbb{R}$ has a

least upper bound

(3) $\Rightarrow$ f attains ~~the~~ value

$$\text{L.U.B.} \{ f(x) \mid x \in [a, b] \}$$

somewhere

UBC

Math 320 - Real Variables

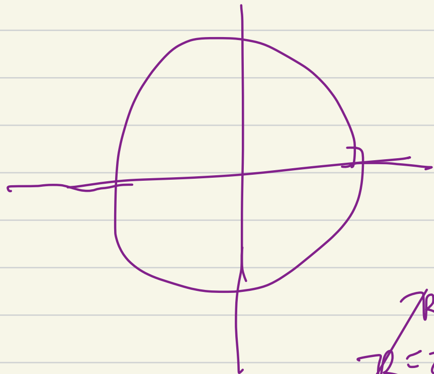
Consequence:

look at

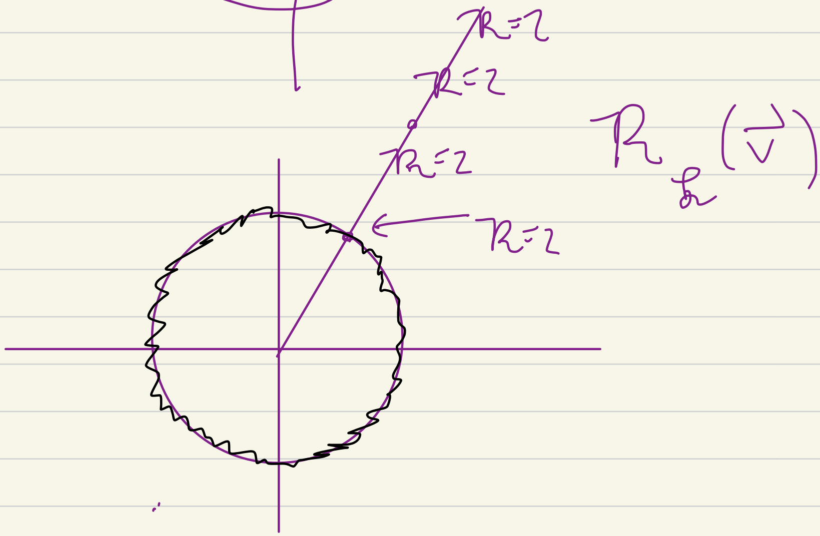
unit sphere in $\mathbb{R}^n$

$$= \{ \vec{x} = (x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 = 1 \}$$

$$\{ \vec{x} \mid \|\vec{x}\|_2 = 1 \}$$



$n=2$



Claim: If f : unit sphere in $\mathbb{R}^n$ $\longrightarrow \mathbb{R}$

and f is continuous, then f

attains its maximum somewhere.

Claim 2: $R = R_{f,(\,,\,)}$ is continuous

So there is a $\vec{v} \in$ unit sphere in $\mathbb{R}^n$ s.t.

$$R_{\mathcal{L}}(\vec{v}) \geq R_{\mathcal{L}}(\vec{v}\,') .$$

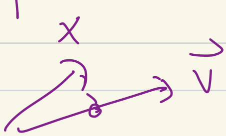
Claim: $\mathcal{L}\vec{v} = \lambda \vec{v}$, $\lambda = R_{\mathcal{L}}(\vec{v})$.

(1) $\mathcal{L}\vec{v} = c\vec{v} + \vec{w}$, $(\vec{w}, \vec{v}) = 0$,

in fact $\vec{x} \in \mathbb{R}^n$,

$$\vec{x} = c\vec{v} + \vec{w}, \quad (\vec{w}, \vec{v}) = 0$$

why?



Caution! $n=2$

$(\vec{x},\vec{y})_{\vec{\omega}}$ $\vec{\omega} = (3,20)$

$(\vec{x},\vec{y})_{\vec{\omega}} = 3x_1y_1 + 20x_2y_2$

$\|\vec{e}_1\|_{\vec{\omega}} = \left\|\begin{bmatrix}1\\0\end{bmatrix}\right\|_{\vec{\omega}} = \sqrt{\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)} = \sqrt{3}$

$\|\vec{e}_2\|_{\vec{\omega}} = \sqrt{20}$

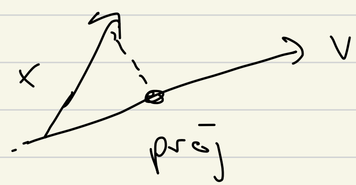
$\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|_{\vec{\omega}} = \sqrt{3+20}$ - - -

Dot product:

$$\mathrm{proj}_{\vec{v}}(\vec{x}) = \vec{v}\ \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}$$

$$\vec{x} = \mathrm{proj}_{\vec{v}}(\vec{x}) + \vec{w}, \qquad \vec{w}\cdot\vec{v} = 0$$

think of



=

$$\mathrm{proj}_{\vec{v},\langle\,,\rangle}(\vec{x}) = \vec{v}\ \frac{\langle\vec{x},\vec{v}\rangle}{\langle\vec{v},\vec{v}\rangle}$$

any dot product

Then

$$\vec{x} = \text{proj} + \vec{w}$$

$$\vec{v}\,\frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})}$$

$$\vec{w} = \vec{x} - \text{proj}$$

$$(\vec{v},\vec{x}) = \left(\vec{v},\ \vec{v}\,\frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})} + \vec{w}\right)$$

$$\|$$

$$(\vec{v},\vec{x}) = \underbrace{\left(\vec{v},\ \vec{v}\,\frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})}\right)}_{(\vec{x},\vec{v})} + (\vec{v},\vec{w})$$

$$\cancel{(\vec{v},\vec{x})} = \cancel{(\vec{x},\vec{v})} + \underbrace{(\vec{v},\vec{w})}_{=0}$$

$\Rightarrow$

$$\mathcal{L}\vec{v} = \underbrace{\text{proj}_{\vec{v}}}_{\substack{\text{proj wrt} \\ \text{the inner} \\ \text{product at} \\ \text{hand}}}(\mathcal{L}\vec{v}) + \vec{w}$$

(like $(\ ,\ )_{\vec{\pi}}$ or $(\ ,\ )_{1/\vec{\pi}}$)

$$= c\vec{v} + \vec{w}$$

$\vec{w}$ ortho to $\vec{v}$, i.e. $(\vec{v}, \vec{w}) = 0$.

$=$

Claim: $\vec{w} = \vec{0}$.

Use "variational argument"

4 minute break

$$\mathcal{L}\vec{v} = c\vec{v} + \vec{w}, \quad (\vec{v},\vec{w}) = 0.$$

Consider $\varepsilon \in \mathbb{R}$ (think of ε near 0)

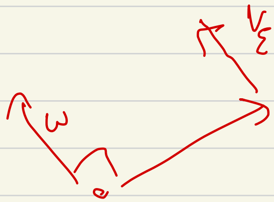
$$\vec{v}_\varepsilon := \vec{v} + \varepsilon\vec{w}.$$

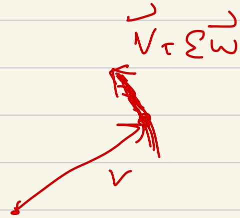
$$(\vec{v}_\varepsilon, \vec{v}_\varepsilon) = (\vec{v} + \varepsilon\vec{w}, \vec{v} + \varepsilon\vec{w})$$

$$= (\vec{v},\vec{v}) + \underbrace{\varepsilon(\vec{w},\vec{v})}_{0} + \underbrace{\varepsilon(\vec{v},\vec{w})}_{0} + \varepsilon^2(\vec{w},\vec{w})$$

$$= (\vec{v}, \vec{v}) + \varepsilon^2 (\vec{\omega}, \vec{\omega})$$

Think of $(\, , \,)$ = dot product





no ε term

Think of $R_{\mathcal{L}}(\vec{v}_\varepsilon) \leq R_{\mathcal{L}}(\vec{v})$

$$\frac{(\mathcal{L}\vec{v}_\varepsilon, \vec{v}_\varepsilon)}{(\vec{v}_\varepsilon, \vec{v}_\varepsilon)} = \frac{\qquad}{(\vec{v}, \vec{v}) + \text{order}(\varepsilon^2)}$$

$$(\mathcal{L}\vec{v}_\varepsilon, \vec{v}_\varepsilon) = (\mathcal{L}(\vec{v}+\varepsilon\vec{w}), \vec{v}+\varepsilon\vec{w})$$

$$= (\mathcal{L}\vec{v},\vec{v}) + \underbrace{\varepsilon(\mathcal{L}\vec{w},\vec{v}) + \varepsilon(\mathcal{L}\vec{v},\vec{w})}$$

self-adj

$$+ \varepsilon^2(\mathcal{L}\vec{w},\vec{w})$$

$$= \underline{(\mathcal{L}\vec{v},\vec{v}) + 2\varepsilon(\mathcal{L}\vec{v},\vec{w})} + \text{order}(\varepsilon^2)$$

$$R_{\mathcal{L}}(\vec{v}_\varepsilon) = \frac{(\mathcal{L}\vec{v},\vec{v}) + 2\varepsilon(\mathcal{L}\vec{v},\vec{w}) + O(\varepsilon^2)}{(\vec{v},\vec{v}) + O(\varepsilon^2)}$$

if $(\mathcal{L}\vec{v},\vec{w}) > 0$

$\Rightarrow$ $\varepsilon > 0$, small

$$R_{\mathcal{L}}(\vec{v}_\varepsilon) = \frac{(\mathcal{L}\vec{v},\vec{v}) + \varepsilon\left(\substack{\text{positive}\\ \text{real}}\right) + O(\varepsilon^2)}{(\vec{v},\vec{v}) + O(\varepsilon^2)}$$

$$\frac{1}{(\vec{v},\vec{v}) + O(\varepsilon^2)} \qquad \varepsilon \text{ small} \qquad = \frac{1}{(\vec{v},\vec{v})} + \text{Order}(\varepsilon^2)$$

$$R_{\mathcal{L}}(\vec{v}_\varepsilon) = \left(\frac{1}{(\vec{v},\vec{v})} + O(\varepsilon^2)\right)\left((\mathcal{L}\vec{v},\vec{v}) + \varepsilon\left(2(\mathcal{L}\vec{v},\vec{w})\right) + O(\varepsilon^2)\right)$$

$\varepsilon > 0$ small

$$> R_{\mathcal{L}}(\vec{v})$$

$$R_{\mathcal{L}}(\vec{v}_\varepsilon) = \frac{(\mathcal{L}\vec{v},\vec{v}) + 2\varepsilon(\mathcal{L}\vec{v},\vec{w})}{(\vec{v},\vec{v})}$$

$+$ term order ε^2

$$= R_{\mathcal{L}}(\vec{v}) + 2\varepsilon \frac{(\mathcal{L}\vec{v},\vec{w})}{(\vec{v},\vec{v})} + O(\varepsilon^2)$$

$\Longrightarrow$ $(\mathcal{L}\vec{v},\vec{w})$ can't be positive

(small $\varepsilon > 0$)

AND

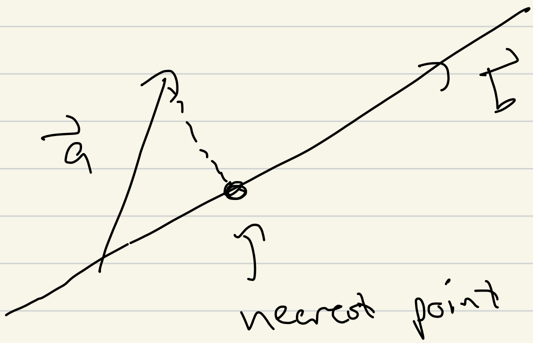
" " " negative

(small $\varepsilon < 0$ near 0)

Rem!

"variational argument"

is done a lot --



$$\min_{c} \left\| \vec{a} - c\vec{b} \pm \varepsilon \vec{b} \right\|$$

Hence

$$(\mathcal{L}\vec{v}, \vec{w}) = 0$$

$$\mathcal{L}\vec{v} = c\vec{v} + \vec{w}$$

$$(c\vec{v} + \vec{w}, \vec{w}) = 0$$

$$c\underbrace{\cancel{(\vec{v}, \vec{w})}}_{0} + \underbrace{(\vec{w}, \vec{w})}_{=0} = 0$$

$$\Rightarrow \vec{w} = \vec{0}$$