CPSC 531F March 9

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- if I don't reply to your Piazza post in 24 hours, please email to

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Subject: $\underbrace{\text{CPSC 531F}}_{\text{somewhere}}$

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Today: - Inner products on $\mathbb{R}^n$,

- orthonormal eigenbases for self-adjoint operators.

Reversible Markov chains:

Irreducible Markov matrix $P \in M_n(\mathbb{R})$,

unique stationary distribution $\vec{\pi} \in \mathbb{R}^n$

($\vec{\pi}$ stochastic, $\vec{\pi}^T P = \vec{\pi}^T$)

reversible:

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall i,j \in [n]$$

detailed balance equations

(e.g. C_3: $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\pi = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$,

$p_{12} = 1$, $p_{21} = 0$ not reversible)

Why reversibility?

(1) it implies "time reversible"

for all i_1, i_2

$$\pi_{i_1} P_{i_1 i_2} = \pi_{i_2} P_{i_2 i_1}, \text{ also}$$

$$\pi_{i_1} P_{i_1 i_2} P_{i_2 i_3} = \pi_{i_3} P_{i_3 i_2} P_{i_2 i_1}, \text{ also}$$

$$\pi_{i_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k}$$

$$= \pi_{i_k} P_{i_k i_{k-1}} \cdots \quad P_{i_2 i_1}$$

"running P forward" $\longleftrightarrow$

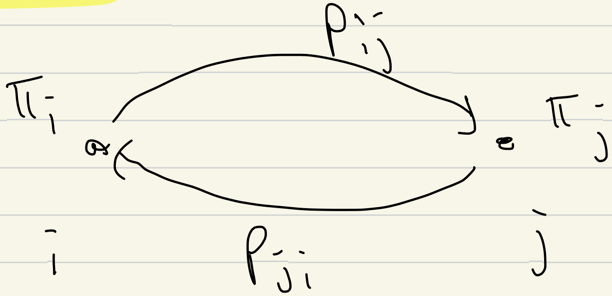
"running P backward"

(w/o talking about "invariant measures $[n]^{\mathbb{Z}}$")

(2) to check

$$\pi_i p_{ij} = \pi_j p_{ji}$$

is "local"



$\forall i, j$

or $$\frac{\pi_i}{\pi_j} = \frac{p_{ji}}{p_{ij}}$$

you might know π_i / π_j, p_{ji}/p_{ij}

without knowing $\pi_i, \pi_j, p_{ij}, p_{ji}$

e.g., Metropolis chain:

- take any irreducible Markov matrix Ψ (following [Levin-Peres])

- take any stochastic $\vec{\pi} \in \mathbb{R}^n$ (it has to have strictly positive entries)

$$\begin{tikzcd} i \arrow[r, bend left, "\psi_{ij}"] & j \arrow[l, bend left, "\psi_{ji}"] \end{tikzcd}$$

form P Metropolis$(\Psi, \vec{\pi})$:

if $i \neq j$:

$$P_{ij} = \psi_{ij} \min\left(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right)$$

claim: $\forall i \in [n]$ $\sum_{j \neq i} p_{ij} < 1$ so we can

Set

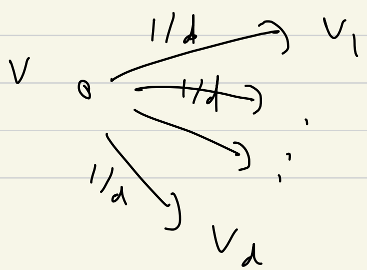
$$p_{ii} = 1 - \sum_{j \neq i} p_{ij}$$

This is reversible and has stationary distribution $\vec{\pi}$.

Intuitively:

Ψ is symmetric, e.g., d-regular graph G

0,1 if no multiple edges

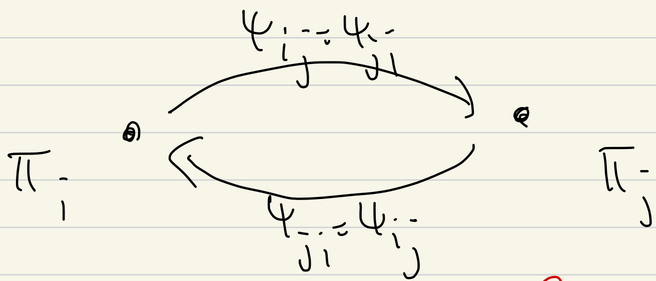


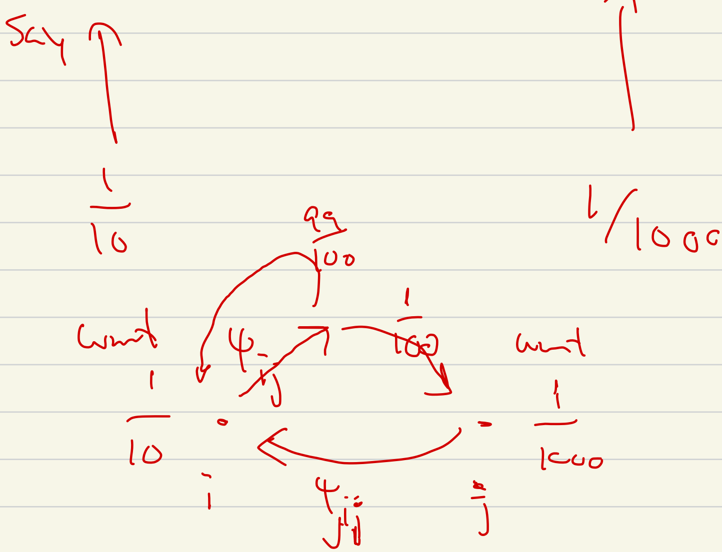
$$P = \frac{1}{d} A_G$$

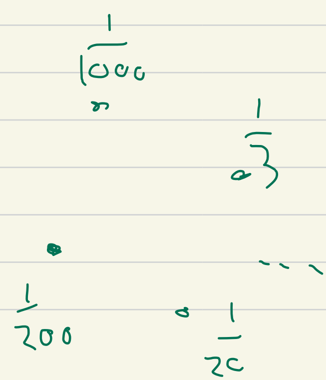
$\psi_{ij} = \psi_{ji}$ for all i, j,

Metropolis(Ψ, π):

$$p_{ij} = \psi_{ij} \min\left(1, \frac{\pi_j}{\pi_i}\right)$$







Typical application!

minimize $f : \bar{V} \to \mathbb{R}$

$\bar{V}$ = all possible positions of $\underbrace{\text{particles}}_{1000}$ in $\underbrace{\text{some box}}_{100{,}000}$

π_i prop. to $e^{-\beta f(i)}$ $\beta > 0$

$f(j) = 100$ more common than

$f(i) = 2$: $\pi_i \sim e^{-2\beta}$ $\pi_j \sim e^{-100\beta}$

If you want

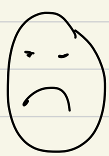
π_i proportional to $e^{-\beta f(i)}$

then

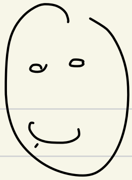
$$\pi_i = \frac{e^{-\beta f(i)}}{\left\{ \sum_{j=1}^{n} e^{-\beta f(j)} \right\}}$$

can't compute this n very large

$n = (100,000)^{1000}$



$\beta = 0$, all π_i prop to 1, all π_i's the same



$$\frac{\pi_i}{\pi_j} = \frac{e^{-\beta f(i)}}{e^{-\beta f(j)}}$$

don't need ("partition function")

$$Z = Z(\beta, f)$$

$$= \sum_{j=1}^{n} e^{-\beta f(j)}$$

to run Metropolis chain ...

So can run " " "locally"

For us:

(3) P has real eigenvalues

and <u>ON</u> eigenbasis where we

introduce more general dot products:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$ usual dot product in $\mathbb{R}^n$

We'll use notation: If $w_1, \ldots, w_n > 0$

$(\vec{x}, \vec{y})_{\vec{w}}$ inner product of $\vec{x}$ and $\vec{y}$ w.r.t. weights $\vec{w} = (w_1, \ldots, w_n)$

given by

$$\boxed{(\vec{x},\vec{y})_{\vec{w}} = x_1 y_1 w_1 + \dots + x_n y_n w_n}$$

More generally an <u>inner product</u> on $\mathbb{R}^n$ is a rule $\vec{x},\vec{y} \mapsto (\vec{x},\vec{y})$

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

s.t.

(1) $(\vec{x},\vec{x}) \geq 0$, equality iff $\vec{x}=\vec{0}$

(2) $(\vec{x},\vec{y}) = (\vec{y},\vec{x})$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$

(3) for $\alpha_1,\alpha_2 \in \mathbb{R}$, $\vec{x}_1,\vec{x}_2 \in \mathbb{R}^n$

$$(\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2, \vec{y}) = \alpha_1(\vec{x}_1,\vec{y}) + \alpha_2(\vec{x}_2,\vec{y}).$$

(2) & (3)

$$\Rightarrow \quad (\vec{x}, \beta_1 \vec{y}_1 + \beta_2 \vec{y}_2) = \beta_1 (\vec{x}, \vec{y}_1) + \beta_2 (\vec{x}, \vec{y}_2)$$

$\Rightarrow$ etc.

Df $\quad \|\vec{x}\| = \left((\vec{x}, \vec{x}) \right)^{1/2}$

here $\| \ \|$ is always w.r.t. $(\ , \ )$ inner product

$$\left| (\vec{x}, \vec{y}) \right| \leq \|\vec{x}\| \, \|\vec{y}\|$$

gives $\quad \cos \vartheta := \dfrac{(\vec{x}, \vec{y})}{\|\vec{x}\| \, \|\vec{y}\|}$

We say that $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$

linear transformation (i.e. $\vec{x} \to A\vec{x}$

for some $A \in M_n(\mathbb{R})$) on

an inner product space

(here $\mathbb{R}^n$ for some n, plus given

inner product $(\, , \,)$) is

self-adjoint if

$$(\mathcal{L}\vec{x}, \vec{y}) = (\vec{x}, \mathcal{L}\vec{y}) \quad (*)$$

e.g.,

$$(\vec{x}, \vec{y}) = \text{usual dot product} = \vec{x} \cdot \vec{y}$$

$$= x_1 y_1 + \ldots + x_n y_n$$

then if

$\mathcal{L}$ given by $x \longmapsto Ax$,

$\mathcal{L}$ (or A) is self-adjoint

iff $\quad A = A^T$.

$$(*) \Leftrightarrow \quad (A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y})$$

e.g.

$$P = \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}, \qquad \pi = \left(\frac{2}{3}, \frac{1}{3}\right)$$

$$P \neq P^T$$

Claim!

① $(P\vec{x}, \vec{y})_{\vec{\pi}} = (\vec{x}, P\vec{y})_{\vec{\pi}}$

② $(\vec{x}^T P, \vec{y}^T)_{1/\vec{\pi}}$ $\qquad 1/\vec{\pi} = \left(\frac{3}{2}, \frac{3}{1}\right)$

$$= (\vec{x}^T, \vec{y}^T P)_{1/\vec{\pi}}$$

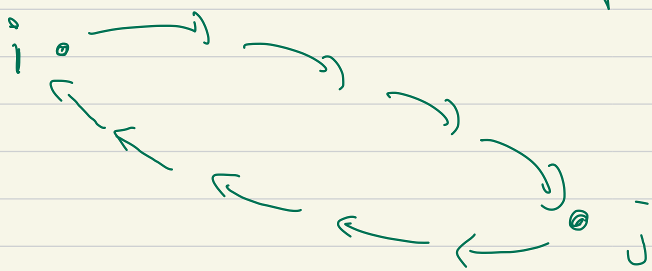
More generally, if P is reversible,

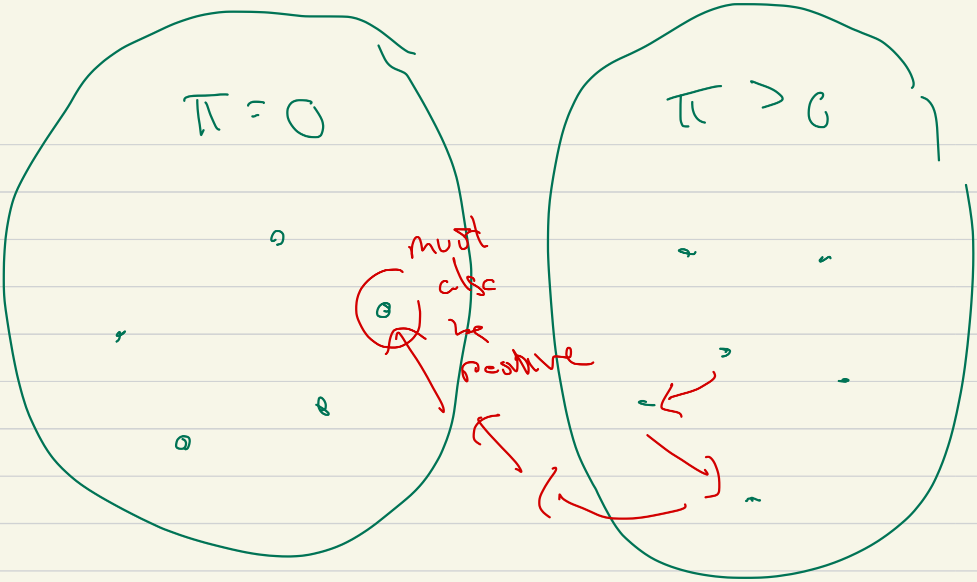
① $$(P\vec{x}, \vec{y})_{\vec{\pi}} = (\vec{x}, P\vec{y})_{\vec{\pi}}$$

② $$(\vec{x}^T P, \vec{y}^T)_{1/\vec{\pi}} = (\vec{x}^T, \vec{y}^T P)_{1/\vec{\pi}}.$$

P irreducible $\Rightarrow$ $\vec{\pi}$ will have all pos components

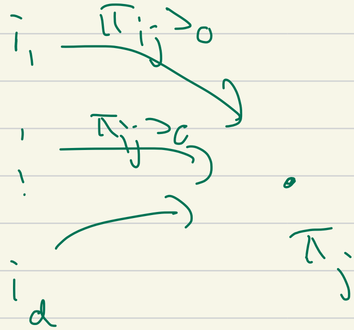
$\pi_i > 0$,





$$\vec{\pi}^T P = \vec{\pi}^T$$

$$\sum_i \pi_i P_{ij} = \pi_j$$



Break 4 minutes

Evidence!

$$P = \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}$$

$$P - I = \begin{bmatrix} -.01 & .01 \\ .02 & -.02 \end{bmatrix}$$

$$P - (.97)I = \begin{bmatrix} .02 & .01 \\ .02 & .01 \end{bmatrix}$$

eigenvectors!

P column vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1 \end{bmatrix} \neq 0$$

$$\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1/2\\-1\end{bmatrix} \right)_{\pi} = 0$$

$$= x_1 y_1 \pi_1 + x_2 y_2 \pi_2$$

$$= 1\cdot\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right) + 1(-1)\left(\tfrac{1}{3}\right) = 0 \text{ !!}$$

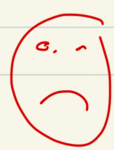
row eigenvectors!

$\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\end{bmatrix}$ $\lambda = 1$

$\begin{bmatrix}1 & -1\end{bmatrix}$ $\lambda = .97$

viewing $\circ$ as applying to row vectors

$$\left(\tfrac{2}{3}\ \ \tfrac{1}{3}\right) \circ \left(1\ \ -1\right) \neq 0$$



$$\left(\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & -1 \end{bmatrix} \right)_{\frac{1}{\frac{1}{\pi}}}$$

$$= \left(\qquad , \qquad \right)_{\left[\frac{3}{2}, \frac{3}{1}\right]}$$

$$= \frac{2}{3} \cdot 1 \cdot \frac{3}{2} + \frac{1}{3}(-1)\frac{3}{1} = 0 \quad !!$$

Rayleigh quotient:

$$R_{\mathcal{L}}(\vec{v}) := \frac{(\mathcal{L}\vec{v}, \vec{v})}{(\vec{v}, \vec{v})}$$

What is

$$\max_{\vec{v} \neq 0} R_A(\vec{v}) \quad ??$$

Det, pred, sym mat A

$$\max_{\vec{v} \neq 0} \frac{A\vec{v} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Class ended

$$\begin{tikzcd} i \arrow[r, bend left] & j \arrow[l, bend left] \end{tikzcd}$$

$\pi(i)$ $\pi(j)$

simplest! $\psi_{ij} = \psi_{ji}$, $\Psi = \Psi^T$

favor i's with $\pi(i)$ large

(psi $= \Psi$

phi $= \Phi$)

$$\Psi_\alpha = \Psi_1 \alpha + \Psi_2 (1-\alpha)$$

$0 \le \alpha \le 1$

Ψ_1

Ψ_2

$$\text{Metrap}(\Psi_\alpha, \vec{\pi}) = \alpha_1 \,\text{Metr}(\Psi_1, \vec{\pi}) + \alpha_2 \,\text{Metr}(\Psi_2, \vec{\pi})$$

$\pi(i) \geq \pi(j)$

$i \qquad j$

Problem: $\Psi_1 = I$ (irr $\alpha < 1$)

$$\Psi_\alpha = \alpha I + (1-\alpha)\Psi_2$$

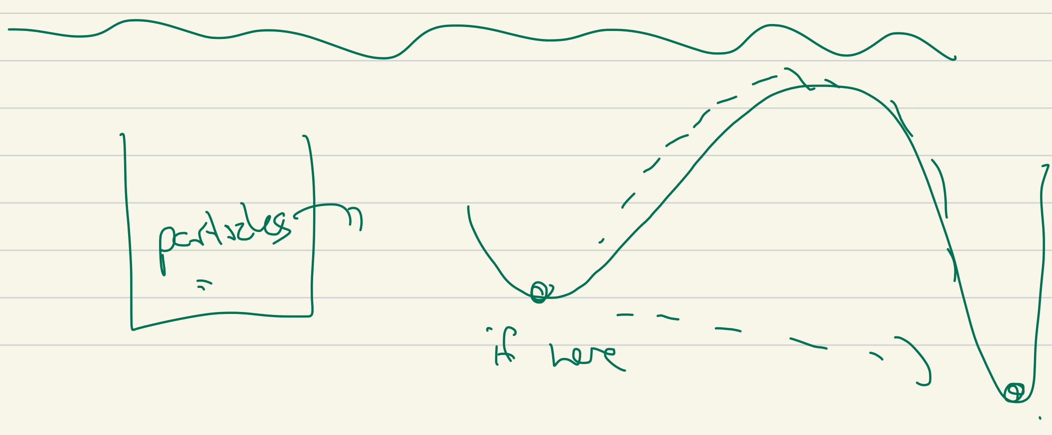
$$\text{Metr}(\ , \pi) = \alpha I + (1-\alpha)\,\text{Metr}(\Psi_2, \pi)$$

$i \longrightarrow j \qquad j \neq i$

$\pi(i) \qquad \pi(j)$

$$p_{ii} = 1 - \underbrace{\text{others}}_{\sum_{j \neq i} p_{ij}}$$

$$\text{Metro}(\Psi_{.99999}, \vec{\pi}) = (.99999) \text{ if } \ldots$$
$$(.00001) \text{ else}$$



$\rho(x)$, $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$

stochastic

$$\int_{-\infty}^{\infty} \rho(x) = 1$$

$$\rho_{t+\underbrace{dt}}$$

$$\rho_{t+dt}(y) = \quad \rho_t(y) = \left(\int \rho_t(x)\, P(x,y)\, dx \right)$$

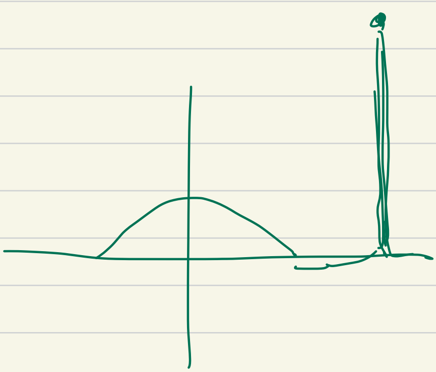
dt

$$u_t = \Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \cdots$$

$$\cancel{\frac{\partial \rho}{\partial t}(y) = \int \rho(x)\, p(x, \cdot)\, dx}$$

infinitesimal op

$$\rho_t = \underset{\text{op}}{\text{local}}\,(\rho)$$



X s.t.

$X = 0$ prob $\frac{1}{2}$

prob $\frac{1}{2}$ something with a density function

$$\pi_i p_{ij} = \pi_j p_{ji}$$

density

density

from

continuous time Markov eq.