March 2:

Question:

$$J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

(*) $$(J_3(\lambda))^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} \\ & \lambda^m & m\lambda^{m-1} \\ & & \lambda^m \end{bmatrix}$$

$\underline{m \in \mathbb{Z}}$ ← \integers

$\underline{\mathbb{N}}$ ← \naturals

Exercise: $(A+B)^m$, A,B commute, $AB=BA$

$$= A^m + \binom{m}{1}A^{m-1}B + \ldots + B^m$$

$$J_3(\lambda) = I + \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = 0$$

(*) can't hold for $\lambda = 0$, $m < 0$

What if $m < 0$, $\lambda \neq 0$ ...

Exercise 3.21 new, higher number

show that (*) holds from m negative integer, $\lambda \neq 0$.

Now, we know, d-regular graph, G,

$$A_G = \frac{d}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + E$$

$\mathcal{E}$: $\|\mathcal{E}\|_2$ L^2-operator norm

$$= \underbrace{\max_{i \geq 2} |\lambda_i(G)|}_{\rho(G)}$$

$\mathcal{E}$ "small" in $\|\mathcal{E}\|_{L^2}$ if ρ small

$$\|\mathcal{E}\|_{L^2} = \max_{\substack{\vec{v} \in \mathbb{R}^n \\ \vec{v} \neq 0}} \frac{\|\mathcal{E}\vec{v}\|_2}{\|\vec{v}\|_2} .$$

$=$

Remark! $\|\mathcal{E}^m\|_{L^2} = \|\mathcal{E}\|_{L^2}^m$

$\mathcal{E}$ symmetric, otherwise $\|\mathcal{E}^m\|_{L^2} \leq \|\mathcal{E}\|_{L^2}^m$

$\rightsquigarrow$ encountered a <u>matrix</u> <u>norm</u> on $M_n(\mathbb{R})$: a norm $\|\cdot\| : M_n(\mathbb{R}) \to \mathbb{R}$

(1) $\| \ \|$ is a norm on the vector space $M_n(\mathbb{R})$:

- $\|A+B\| \leq \|A\| + \|B\|$
- $\|\alpha A\| \leq |\alpha| \, \|A\|$, $\alpha \in \mathbb{R}$
- $\|A\| \geq 0$, equ. iff $A = 0$

<u>(2)</u> $\|AB\| \leq \|A\| \, \|B\|$

So $\|\cdot\|_{L^2}$, or $\|\cdot\|_{L^p}$ any p you get a matrix norm.

Homework: G d-reg graph

comes from $\lambda_1 v_1 v_1^T$

$$A_G = \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \ddots & 1 \end{bmatrix} + \mathcal{E}$$

$\sum_{i=2}^{n} \lambda_i v_i v_i^T$

$$A_G^r = \frac{d^r}{n}\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \ddots & 1 \end{bmatrix} + \mathcal{E}^r$$

$$\|\mathcal{E}\|_{L^2} = \rho := \max_{i \geq 2} |\lambda_i| ,$$

$\rightsquigarrow$ gives information about the mixing time.

Here

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T \longrightarrow$$

$$A^r = \sum_{i=1}^{n} \lambda_i^r v_i v_i^T$$

$$\left(A v_i = \lambda_i v_i \Rightarrow A^r v_i = \lambda_i^r v_i \right)$$

Also ε "lives" on $\vec{1}^{\perp}$!

$$\varepsilon \vec{1} = 0$$

Go Back to

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^m \quad m \text{ negative}$$

$= ??$

Now we have a tool:

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1/\lambda & 0 \\ 0 & 1 & 1/\lambda \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda \neq 0$

$$= \lambda \left[\underbrace{I}_{} + \underbrace{N}_{\downarrow} \right]$$

$$N = \begin{bmatrix} 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \\ 0 & 0 & 0 \end{bmatrix}$$

N nilpotent!

$$N^{\text{some power}} = 0$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

if $|x| < 1$

$=$

Also

$$(I - A)^{-1} = I + A + A^2 + \cdots$$

holds for ... which A ??

$$\left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} d_1 & c & c \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} + \begin{bmatrix} d_1^2 & & \\ & d_2^2 & \\ & & d_3^2 \end{bmatrix} + \cdots$$

Holds if $|d_1|, |d_2|, |d_3| < 1$

What about

$$(I-A)^{-1} = I + A + A^2 + \dots$$

$$\uparrow$$

$\mathrm{diag}(d_1, d_2, d_3)$ OK if $|d_1|, |d_2|, |d_3| < 1$

What about

$$(I-N)^{-1} = I + N + N^2 + \dots$$

if $N^r = 0$ for some $r \in \mathbb{N}$

Claim:

$$(I - A) = I + A + A^2 + \cdots$$

provided that

intuitive: " $A^m + A^{m+1} + \cdots \xrightarrow{\text{as } m \to \infty} 0$ "

provided that

rigorous: $\|A^r\|_{L^2} < 1$ for some r,

Hence, if $N^r = 0$ some $r \in \mathbb{N}$,

then $(I - N)^{-1} = I + N + N^2 + \cdots + N^{r-1}$

So

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

said

$$(I - A)^{-1} = I + A + A^2 + \ldots$$

if $\|A^r\|_{L^2} < 1$ for some r

$$\Rightarrow \left(I - \begin{bmatrix} 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1}$$ we can using

$$= \left(\frac{1}{1-x}\right)^2 = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + \ldots\right)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$=$

$$\left(\frac{1}{1-x}\right)^m = \left(\frac{d}{dx}\right)^{m-1}\left(\frac{1}{1-x}\right) = \text{etc.}$$

$$(I - A)^m = \text{some power series}$$

$\Rightarrow$ allow us to compute $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^{-m}$

$m \in \mathbb{N}$.

Goal is to talk about ε-mixing time and mixing time of Markov chains.

=

We take an irreducible Markov matrix $P \in M_n(\mathbb{R})$; P has a unique stationary distribution $\pi \in \mathbb{R}^n$ stochastic

$$\pi^T P = \pi^T$$

=

1

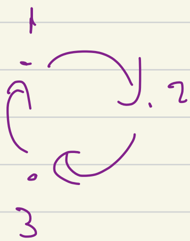
$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Claim!

$$\lim_{m\to\infty} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}^m = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$=$

(2)



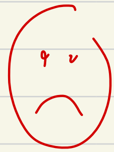
$$P = C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\pi^T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}}_{\text{eigenvector!}} C_3 = \underbrace{\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}}$$

eigenvector! $\pi^T C_3 = \pi^T \cdot 1$

$P = C_3$: $P^m = C_3^m$



$$\xrightarrow[m \to \infty]{?} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

ε- mixing time = ??

(1) $$\pi = \lim_{m \to \infty} \frac{I + P + \dots + P^m}{m+1}$$

$$= \lim_{m \to \infty} \text{Avg of} \left(P^0, P^1, \dots, P^m \right)$$

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_3^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_3^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② Say that P is aperiodic,

i.e. if you take the GCD

of all $k \in \mathbb{N}$ s.t. $(P^k)_{ii} > 0$

for some i equals 1, then

$$\lim_{m \to \infty} P^m = \begin{bmatrix} - & \pi & - \\ - & \pi & - \\ - & \pi & - \end{bmatrix} = \mathbb{1} \cdot \pi^T$$

$$= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \pi^T$$

$=$

e.g. $(C_3^k)_{ii} > 0$ then k must be

a multiple of 3

so "period" of C_3 is 3

More generally

period(P) = GCD all $k \in \mathbb{N}$

s.t. (P^k) has a

non-zero diagonal

entry

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4 minute break

Mixing time:

Irreducible Markov matrix, P,

we set

$$d(t) := \max_{i=1,\dots,n} \| \vec{e}_i P^t - \pi \|_{TV}$$

max distance to stationary after t steps.

$$\bar{d}(t) := \max_{i,j} \| \vec{e}_i P^t - \vec{e}_j P^t \|$$

① Thm: $\bar{d}(t+s) \leq \bar{d}(t)\bar{d}(s)$

② Thm: $d(t) \leq \bar{d}(t) \leq 2\, d(t)$

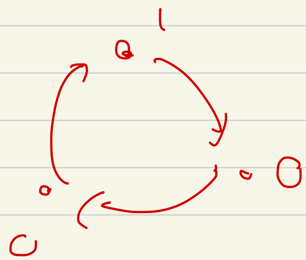
The ε-mixing time:

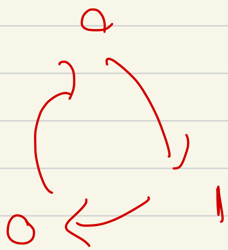
$$t(\varepsilon) = \min_{t \in \mathbb{N}} \left(d(t) \leq \varepsilon \right)$$

Example:

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Big\| \underbrace{\vec{e}_1 C_3^t - \vec{e}_2 C^t}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \Big\| = \sqrt{2}$$

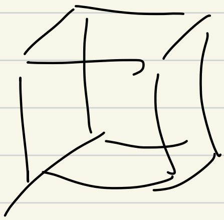




S_6

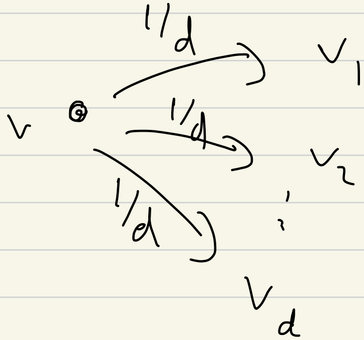
$$t(\varepsilon) = \text{"}+\infty\text{"}$$

for ε small



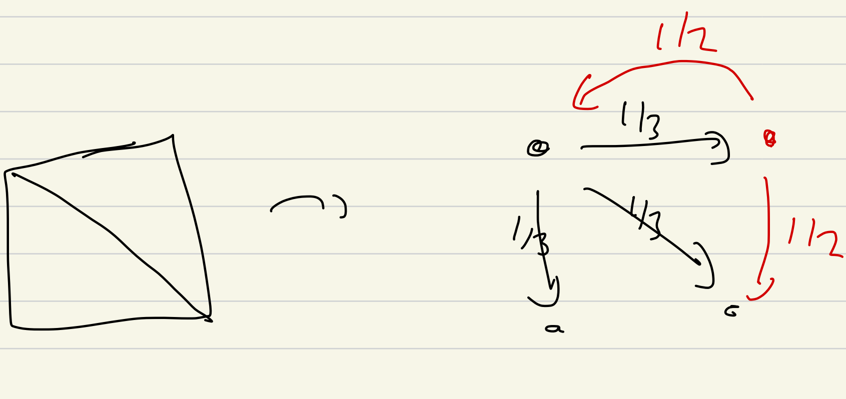
Graph $\mathbb{B}^3$

Graph, G:

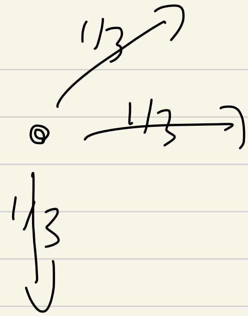


Markov chain

$d = \deg(v)$



$\mathbb{B}^3$



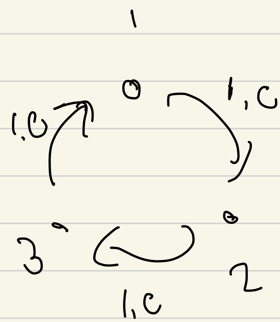
periodic, aperiodic,

mixing times finite or infinite?

? ? ? ? ? ?

Class ends ~

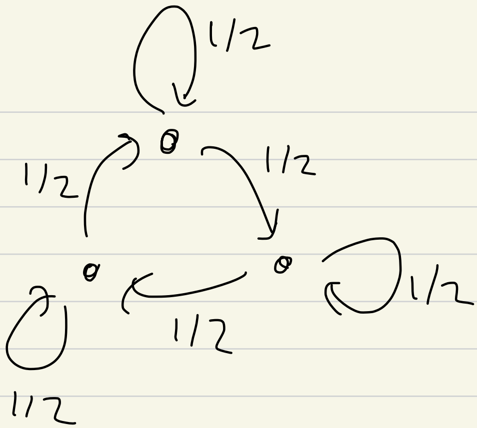
$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} ;$$



$$\operatorname{avg}\left(C_3^0, C_3^1, \ldots, C_3^m\right)$$

$$\rightarrow \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$C_3^m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



$$P_{new} = \underbrace{\tfrac{1}{2} I}_{\text{“lazy”}} + \tfrac{1}{2} C_3$$

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = P_{new}$$

$$\vec{e}_1 P_{new}^{m} \xrightarrow[m \to \infty]{} \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\vec{e}_1 \hat{P}^t = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$t = 1$ (!)

ε-mixing time $= 1$ for any ε (!)

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \end{tikzcd}$$

$$\hat{P} = \begin{bmatrix} .01 & .99 \\ .99 & .01 \end{bmatrix}$$

lazy: add self-loop with prob p.

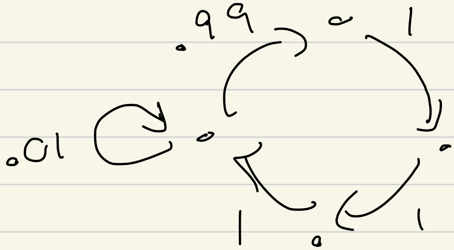
$$\begin{tikzcd} \bullet \arrow[loop left, ".01"] \arrow[r, bend left, ".99"] & \bullet \arrow[l, bend left, ".99"] \arrow[loop right, ".01"] \end{tikzcd}$$

$$\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}$$

$$\begin{tikzcd} \bullet \arrow[loop left, ".99"] \arrow[r, bend left, ".01"] & \bullet \arrow[loop right, ".98"] \arrow[l, bend left, ".02"] \end{tikzcd}$$

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$ mixing time 1

$$P = \begin{bmatrix} .499 & .501 \\ .499 & .501 \end{bmatrix} \text{ or } \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \text{small}$$



period = 1