CPSC 531F Feb 23

- Homework 1: hand in by end of this week. email to jf@cs.ubc.ca

Subject: CPSC 531F Homework 1

Last time: "Expander Mixing Lemma"

We'll also talk about

- Reversible Markov chains/matrices
- SVD (Singular Value Decomposition)
- Expanders

Symmetric, or more generally self-adjoint matrices

For now: we assume that if A is symmetric, $A \in M_n(\mathbb{R})$, then A has ① real eigenvalues ② an ON (orthonormal) eigenbasis.

$$\begin{cases} A\vec{v}_i = \lambda_i \vec{v}_i, & i = 1, \dots, n \\ \lambda_i \text{ real}, & \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} A \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \\ AQ = QD, \quad Q, D \in M_n(\mathbb{R}) \end{cases}$$

D = diagonal, Q orthogonal matrix

Q orthogonal: any of these equivalent cond.

- Q's cols are ON
- Q's row " "
- $QQ^T = I$ - $Q^T = Q^{-1}$ - $Q^TQ = I$
- Q preserves dot product!

  $\vec{v} \longmapsto Q\vec{v}$ preserves dot prods

  $$\vec{v} \cdot \vec{w} = (Q\vec{v}) \cdot (Q\vec{w})$$

- Q preserves angles and lengths

Tool: Rayleigh quotients

$$AQ = QD \iff A = QDQ^{-1}$$

$\rightsquigarrow$ Q, Q^{-1} preserve dot products
(length, angles)

$\rightsquigarrow$ mostly A can be viewed as diagonal matrix, if you are working with dot products, lengths, ...

$$R_A(\vec{v}) := \frac{(A\vec{v}) \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Last time:

$$\lambda_n(A) \|\vec{v}\|^2 \leq (A\vec{v}) \cdot \vec{v} \leq \lambda_1(A) \|\vec{v}\|^2$$

$$\lambda_n \vec{v} \cdot \vec{v} \leq A\vec{v} \cdot \vec{v} \leq \lambda_1 \vec{v} \cdot \vec{v}$$

$$\Rightarrow \quad \lambda_n \leq R_A(\vec{v}) \leq \lambda_1$$

Also $\qquad\qquad\qquad A\vec{v}_i = \lambda_i \vec{v}_i$

$$R_A(\vec{v}_i) =$$

$$= \frac{(A\vec{v}_i)\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i} = \frac{\lambda_i\vec{v}_i\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}$$

$$= \lambda_i$$

$$\left|(A\vec{v})\cdot(\vec{w})\right| \le \left(\max_{1\le i\le n}|\lambda_i|\right)\|\vec{v}\|\,\|\vec{w}\|$$

now we will apply this ...

Remark: If $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$,

then

$$\boxed{D\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}} = \left\{ \begin{array}{l} \text{numerator precursor to} \\ R_D(\vec{u}) \end{array} \right.$$

$$\begin{bmatrix} d_1 u_1 \\ \vdots \\ d_n u_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$= d_1 u_1^2 + d_2 u_2^2 + \dots + d_n u_n^2$$

$$D\vec{e}_1 \cdot \vec{e}_1 = d_1, \quad D\vec{e}_n \cdot \vec{e}_n = d_n$$

In graph theory: G, $V_G = [n]$

so that $A_G \in M_n(\mathbb{R})$

$U, W \subset [n] = \{1, \dots, n\} \leftarrow$ Vertex set of G

then

e_U = indicator function of U $= \sum_{i \in U} e_i$

e_W = — — — — — W $= \sum_{i \in W} e_i$

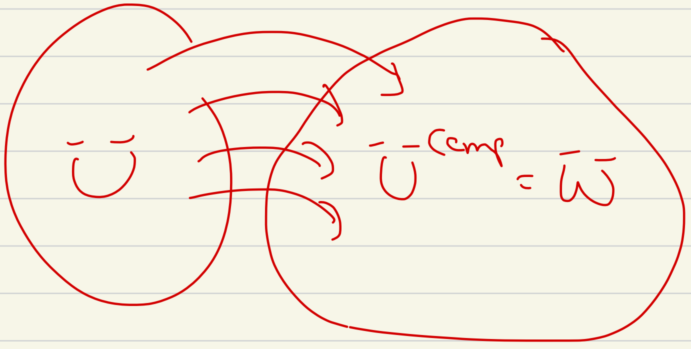
$$e_U^T A_G e_W$$ = # edges running from U to W in G

since A_G is symmetric,

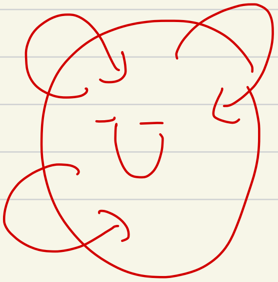
$$e_{\bar{U}}^T A_G e_{\bar{W}} = A_G e_{\bar{U}} \cdot e_{\bar{W}}$$

$$= e_{\bar{U}} \cdot A e_{\bar{W}}$$

Special Case $\bar{W} = \bar{U}^{comp}$



equivalent, if A is d-regular



$$= (e_{\bar{U}}) \cdot (A_G e_{\bar{U}})$$

$$= (A e_{\bar{U}}) \cdot e_{\bar{U}}$$

$$\bar{U}^{comp} = V_G \setminus \bar{U}$$

$$= \{ i \in [n] \mid i \notin \bar{U} \}$$

G is d-regular!

$$e_{\bar{U}}^T A_G \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = d \cdot |\bar{U}|$$

$$e_U^T A_G \left(e_{\bar{U}} + e_{\bar{U}^{comp}} \right)$$

$$e_U^T A_G e_U + e_U^T A_G e_{\bar{U}^{comp}} = d \cdot |\bar{U}|$$

Last time: for any graph

$$A_G \stackrel{?}{=} \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T \quad (*)$$

(where $\lambda_1, \ldots, \lambda_n$ eigenvalues, $\vec{v}_1, \ldots, \vec{v}_n$ corresponding ON eigenbasis)

$$A_G \vec{v}_j \stackrel{?}{=} \left(\sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T \right) \vec{v}_j$$

$$= \sum \lambda_i \vec{v}_i \underbrace{\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}_{\text{ON basis}}$$

$$= \lambda_j \vec{v}_j$$

$(*)$ works on $\vec{v}_j$.

$$A_G \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = d \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{if } G \text{ is } d\text{-regular}$$

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix}, \quad \lambda_1 = d$$

We can take $\vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$ ON eigenbasis.

$$-d \le \lambda_n \le \dots \le \underbrace{\lambda_2 \le \lambda_1}_{\substack{\lambda_2 < \lambda_1 \\ \text{if } G \text{ is connected}}} = d$$

$$A_G = \lambda_1 v_1 \vec{v}_1^T + \underbrace{\lambda_2 \vec{v}_2 \vec{v}_2^T + \dots + \lambda_n \vec{v}_n \vec{v}_n^T}_{\downarrow}$$

$$= \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \underbrace{\mathcal{E}}_{\text{"lives on } \vec{1}^{\perp}\text{"}}$$

① $\mathcal{E}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0$: since

$$\vec{v}_i^{\,T}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \text{ for } i \geq 2$$

$$\mathcal{E} = \left(\lambda_2 \vec{v}_2 \vec{v}_2^{\,T} + \dots + \lambda_n \vec{v}_n \vec{v}_n^{\,T}\right)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \lambda_2 \vec{v}_2 \underbrace{\left(\vec{v}_2^{\,T}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right)}_{0} + \underbrace{\dots}_{0}$$

eigenvalues of $\mathcal{E}$ on all of $\mathbb{R}^n$

$\lambda_2, \dots, \lambda_n$, and $0 \leftrightarrow$ eigenvector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$=$

$$\vec{e}_{\bar{U}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

1's on $\bar{U}$ positions

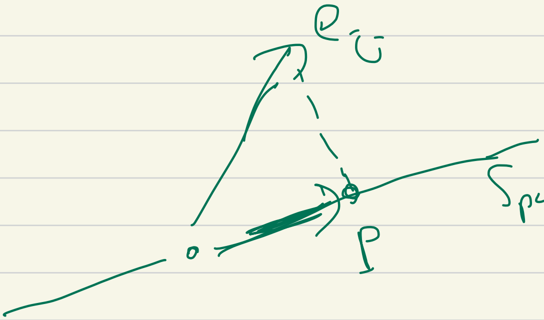
0's elsewhere

$$\operatorname{proj}_{\vec{v}_1}(\vec{e}_{\bar{U}}) = (\vec{e}_{\bar{U}} \cdot \vec{v}_1) \cdot \vec{v}_1$$

$$= \left(\sum_{u \in \bar{U}} \frac{1}{\sqrt{n}} \right) \begin{bmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix}$$

$$= \frac{|\bar{U}|}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\vec{e}_{u} - \overbrace{\text{proj}_{\vec{v}_1} \vec{e}_u}^{P} = \text{leftover}$$



$$\text{Span}(\vec{v}_1) = \text{Span}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}$$

$$\|\text{leftover}\| = \left\| \begin{bmatrix} \\ \\ \end{bmatrix} \right\|$$

EXERCISE: this norm is

$$\sqrt{\frac{|\bar{U}|\,(n-|\bar{U}|)}{n}}.$$

$$A_G = \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \\ 1 & & 1 \end{bmatrix} + \mathcal{E}$$

$$A_G \vec{e}_{\bar{w}} = \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & & 1 \end{bmatrix}(\text{proj})$$

$$+ \ \mathcal{E}(\text{leftover})$$

$$= \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & & 1 \end{bmatrix}\left(\frac{|\bar{w}|}{n}\right)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$+ \ \mathcal{E}(\text{leftover})$$

$$= \frac{d\,|\bar{w}|}{n}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \mathcal{E}(\text{leftover}_{\bar{w}})$$

$$\vec{e}_{U}^{\,T} A_G \vec{e}_W =$$

$$\vec{e}_{U}^{\,T} \quad \frac{d|\overline{W}|}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + (\text{leftover}_{\bar{U}})^T \cdot \mathcal{E} \cdot (\text{leftover}_{W})$$

$$= \frac{d|\overline{W}||\overline{U}|}{n} + \underbrace{(\text{left}_{\bar{U}})^T \mathcal{E} (\text{left}_{W})}$$

$$\left| \text{leftover}_{\bar{U}}^{\,T} \, \mathcal{E} \, \text{left}_{\bar{W}} \right| \le \left(\max_{i \ge 2} |\lambda_i|\right) \| \quad \| \| \quad \|$$

$$\le \left(\max_{i \ge 2}\right) |\lambda_i| \sqrt{\frac{|\overline{W}|(n-|W|)}{n}} \sqrt{\frac{|U|(n-|U|)}{n}}$$

Thm: for any $\bar{U}, \bar{W}$,

edges $\bar{U}$ to $\bar{W}$ (in d-reg G)

$=$

$e_{\bar{U}}^{T} A_G e_W =$

$$\frac{d|W||\bar{U}|}{n} + \text{extra}_{\bar{U},\bar{W}}$$

$$|\text{extra}_{\bar{U},W}| \le \rho \sqrt{\frac{|\bar{W}|(n-\bar{W})}{n}} \sqrt{\frac{|U|(n-|U|)}{n}}$$

$$\le \rho \sqrt{|W|} \sqrt{|U|}$$

$$\rho = \max_{i \ge 2} |\lambda_i|$$ "mixing lemma"

Four minute break

d-regular graph, A_G symmetric

$$\underbrace{\lambda_n \leq \dots \leq \lambda_2}_{\text{if these are small, then}} \leq \lambda_1$$

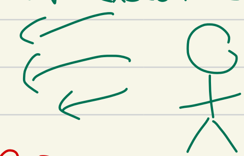
tells a lot of the picture

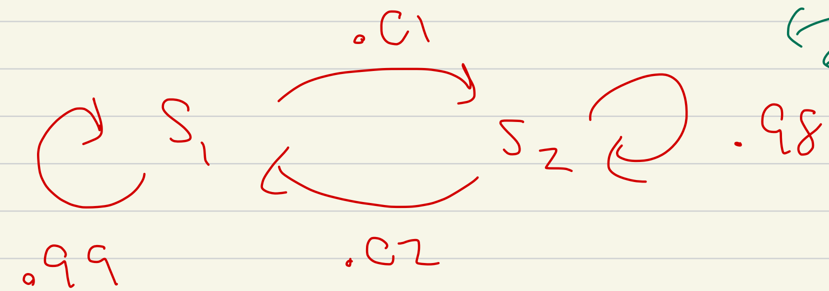
$$\overbrace{\frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \\ 1 & & 1 \end{bmatrix}}^{} + \varepsilon$$

Markov chains!

$$\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} = P \qquad \text{Markov matrix}$$

TV executive





P does not look symmetric ...

<u>Claim:</u> P is symmetric, from a certain point of view..

① "Refinement"

Thought experiment!

$S_1 \qquad S_2$

Dolphin

S_1'

$S_1'' \qquad S_2$

really 2 states

Dolphin sees S_1

$$\begin{tikzcd} & S_1' \\ S_1 \arrow[ur] \arrow[dr] & \\ & S_1'' \end{tikzcd}$$

Dolphin sees: same Markov chain

$$\begin{tikzcd} S_1' \arrow[r, bend left, ".01"] & S_1'' \arrow[l, bend left, ".01"] \end{tikzcd}$$

$$\begin{array}{c|ccc} & S_1' & S_1'' & S_2 \\ \hline S_1' & .98 & .01 & .01 \\ S_1'' & .01 & .98 & .01 \\ S_2 & .01 & .01 & .98 \end{array}$$

} symmetric

Dolphin $S_2 \xrightarrow{.01} S_1'$

$S_2 \xrightarrow[.01]{} S_1''$

instead of

$$S_2 \xrightarrow{.02} S_1$$

Class ends