CPSC 531F, Feb 11

- Lectures recorded available in
  Canvas - Media Gallery ($\leq$ 4G)
  - Zoom cloud ($\leq$ a bit less)

Homework & Supplemental Notes:

Yesterday's §4.3 $\rightarrow$ §4.4

New section 4.3 on Rayleigh quotients:

Thm: Let A be symmetric (Hermitian).

Then for $\vec{w} \in \mathbb{R}^n, \mathbb{C}^n$ (A is $n \times n$)

(1)

$$\lambda_n \|w\|_2^2 \leq \vec{w} \cdot (A\vec{w}) \leq \lambda_1 \|w\|_2^2$$

where A's eigenvalues $\lambda_n \leq \dots \leq \lambda_1$

[equality when $A\vec{w} = \lambda_1 \vec{w}$, $A\vec{w} = \lambda_n \vec{w}$, for right, left inequalities].

(2) $$\underbrace{\|A\|_2}_{} = \max_{\vec{w} \neq 0} \frac{\|A\vec{w}\|_2}{\|\vec{w}\|_2}$$

L^2 operator norm

= max amount that A "streches" the L^2-norm

Cauchy -Schwartz

$$\|A\|_2 = \left(\max_{i=1,\dots,n} (|\lambda_i|) \right)$$

(3) for any $\vec{u}, \vec{w} \in \mathbb{R}^n$ (or $\mathbb{C}^n$)

$$|\vec{u} \cdot A\vec{w}| \leq \left(\max_{i=1,\dots,n} |\lambda_i|\right) \|\vec{u}\|_2 \|\vec{w}\|_2$$

$\boxed{\text{A symmetric: } A \in M_n(\mathbb{R}),\ A = A^T}$ GRAPH THEORY

A Hermitian: $A \in M_n(\mathbb{C}),\ A = A^H$

Notation [H-J]: $A = A^*$ $* = T, H$

What is this saying?

$$A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = \mathrm{diag}(d_1, \ldots, d_n)$$

A symmetric, $d_1, \ldots, d_n \in \mathbb{R}$

$A = A^H$ $\quad d_1, \ldots, d_n : \overline{d_i} = d_i, \; d_i \in \mathbb{R}$

$$\vec{w} \cdot (A\vec{w})$$

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \cdot \begin{bmatrix} d_1 w_1 \\ \vdots \\ d_n w_n \end{bmatrix}$$

$$= d_1 w_1^2 + d_2 w_2^2 + \ldots + d_n w_n^2$$

① A's eigenvalues are $d_1, \dots, d_n$

$$\begin{array}{ccccccc} d_n & \le & \dots & \le & d_2 & \le & d_1 \\ \parallel & & & & \parallel & & \parallel \\ \lambda_n & \le & \dots & & \lambda_2 & \le & \lambda_1 \end{array}$$

Eigenvectors of A are $\vec{e}_1, \dots, \vec{e}_n$

standard basis vectors

$$\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ d_2 \\ 0 \end{bmatrix} = d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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②

$$d_1 w_1^2 + \dots + d_n w_n^2 \le d_1 w_1^2 + d_1 w_2^2 + \dots + d_1 w_n^2$$

$$= d_1 \underbrace{(w_1^2 + w_2^2 + \dots + w_n^2)}$$

$$\vec{w} \cdot A\vec{w} \le d_1 \|\vec{w}\|_2^2 = d_1 \, \vec{w} \cdot \vec{w}$$

furthermore: in

$$\vec{w} \cdot A\vec{w} \leq d_1 \, \vec{w} \cdot \vec{w}$$

equality holds iff whenever

$w_j \neq 0 \Rightarrow d_j = d_1 = \lambda_1$

=

Similarly

$$\lambda_n \|w_1\|_2^2 \leq \vec{w} \cdot A\vec{w} \leq \lambda_1 \|w_1\|_2^2$$

$$\lambda_1 \, w_1 \cdot w_1$$

Similarly

$$\|A\vec{w}\|_2^2 = \left\| \begin{bmatrix} d_1 & & \\ & \ddots & \end{bmatrix} \vec{w} \right\|_2^2 = \left\| \begin{bmatrix} d_1 w_1 \\ \vdots \\ d_n w_n \end{bmatrix} \right\|_2^2$$

$$= d_1^2 w_1^2 + \dots + d_n^2 w_n^2$$

$$\leq \left(\max_i d_i^2\right) \|\vec{w}\|_2^2$$

$$\|A\vec{w}\|_2^2 \leq \left(\max_i d_i^2\right) \|\vec{w}\|_2^2$$

$$\|A\vec{w}\|_2 \leq \left(\max_{i=1,\dots,n} |d_i|\right) \|\vec{w}\|_2$$

So

$$\|A\|_2 \leq \max_{i=1,\dots,n} |d_i|$$

$$\|A\vec{e}_1\|_2 = \|d_1\vec{e}_1\| = |d_1| \|\vec{e}_1\|_2$$

$$\vdots$$

$$\|A\vec{e}_n\|_2 = |d_n| \|\vec{e}_n\|_2$$

③ Cauchy-Schwarz

$$|\vec{u} \cdot A\vec{w}| \leq \|\vec{u}\|_2 \|A\vec{w}\|_2$$

$$|\vec{a} \cdot \vec{b}| = \|\vec{a}\|_2 \|\vec{b}\|_2 \cos(\vartheta_{\vec{a},\vec{b}})$$

$$\leq \|\vec{a}\|_2 \|\vec{b}\|_2$$

$$|\vec{u} \cdot A\vec{w}| \leq \|\vec{u}\|_2 \|\vec{w}\|_2 \left(\max_{i=1,\dots,n} |d_i|\right)$$

① - ③ measuring "how large is

$$A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} ?"$$

General case follows $\underbrace{\text{immediately}}_{\text{once we get used to properties of } Q}$

once we get used to properties of Q

Why?

$$\vec{w} \cdot A\vec{w} \le \lambda_1 \vec{w} \cdot \vec{w}$$

Claim: A is symmetric, then it has
ON eigenbasis, real eigenvalues,
$AQ = QD$ or $A = QDQ^{-1}$

Q has its columns = eigenbasis vectors

$QQ^T = I$ $\quad Q$ = orthogonal matrix

[Complex case $A = UDU^{-1}$, U unitary, $UU^H = I$]

$$\vec{w} \cdot (D\vec{w}) \le \lambda_1 \vec{w} \cdot \vec{w}$$

$\lambda_1, \dots, \lambda_n$ eigenvalues of D, = eigs of A

$$\text{charpoly}_A(t) = \det(tI - A)$$

$$= \det(tI - QDQ^{-1})$$

$$= \det(Q(tI - D)Q^{-1})$$

$$= \det(Q)\det(tI-D)\det(Q^{-1})$$

$$= \det(tI - D) = \text{charpoly}_D(t)$$

(Same proof if Q any invertible
really! $A \sim B$, $A = MBM^{-1}$
eigs A = eigs of B)

$$\underbrace{(\vec{w} \cdot D\vec{w})}_{\vec{w} \,\cdot\, Q^T \vec{w} Q} \le \lambda_1 \, \vec{w} \cdot \vec{w}$$

$\lambda_1(D) = \lambda_1(A)$

Q = orthogonal

$$\underbrace{\vec{w} \cdot (Q^t \vec{w} Q)} = \vec{w} \cdot \vec{w}$$

$$\vec{w}^T Q^T \vec{w} Q = (Q\vec{w}) \cdot (Q\vec{w})$$

$$= \vec{w} \cdot \vec{w}$$

for any orthogonal matrix Q,

More generally, Q orthogonal

$$(Q\vec{a}) \cdot (Q\vec{b}) = \vec{a} \cdot \vec{b}$$

$$\Downarrow$$

$$\vec{a}^T Q^T Q \vec{b}$$

$$= \vec{a}^T I \vec{b} = \vec{a}^T \vec{b} = \vec{a} \cdot \vec{b}$$

Immediately! Q ortho then

- Q preserves $\cdot$

So is

$$\lambda_n \|\vec{w}\|^2 \le \vec{w} \cdot (A\vec{w}) \le \lambda_1 \|\vec{w}\|_2^2 \text{ ??}$$

$$A = QDQ^T$$

$$\lambda_n \|Q\vec{w}\|_2^2 \le (Q\vec{w}) \cdot D(Q\vec{w}) \le \lambda_1 \|Q\vec{w}\|_2^2$$

we verified this

$$\vec{w} \rightsquigarrow Q\vec{w}$$

Example! $f : \mathbb{R}^2 \to \mathbb{R}$

$f(\vec{x})$ near 0 $\simeq f(\vec{0}) + \vec{x} \cdot (\nabla f)(\vec{0})$

$+ \ \vec{x} \cdot \left(\text{Hess}(0)\,\vec{x}\right)$

Say $\text{Hess}(0) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$G = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, but $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is simpler

$= I a + C_2 b$

3 min break

Expander mixing lemma

Stronger form: $A_G \in M_n(\mathbb{R})$

symmetric: $I, J \subset [n] = \{1, \dots, n\}$

$$\vec{e}_I \cdot (A_G \vec{e}_J) = \text{\# edges from } I \text{ to } J$$

$$\mathbb{1}_I \cdot (A_G \mathbb{1}_J)$$

$$\vec{e}_I = \mathbb{1}_I = \chi_I = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

1's in location I

$$= \sum_{i \in I} \vec{e}_i$$

Claim: G d-regular

$$\left| \vec{e}_I \cdot A_G \vec{e}_J - \frac{d}{n}|I|\cdot|J| \right|$$

$$\le \left(\max_{i \ge 2} |\lambda_2|\right) \underbrace{\sqrt{\frac{|I|(n-|I|)}{n}}}_{\text{or } \le \sqrt{|I|}} \sqrt{\frac{|J|(n-|J|)}{n}}$$

Proof:

$$\vec{e}_I = \underbrace{\mathrm{proj}_{\vec{1}}(\vec{e}_I)}_{\left(\text{avg value of } \vec{e}_I\right)\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}^T} + \underbrace{\text{leftover}}_{\text{has norm equal to}}$$

Class ended.

Coupling!

MC: Markov matrix $\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}$

$X_0, X_1, X_2, \ldots$

initial dist X_0: $\begin{bmatrix} \text{prob } X_0 \text{ in state 1} & \text{prob } X_0 \text{ in state 2} \end{bmatrix}$
$= \text{Prob}(X_0)$

$$\text{Prob}(X_n) = \text{Prob}(X_0) \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}^n$$

$X_0, X_1, \ldots$: ProbSpace $\longrightarrow \{1, 2\}$

$$\text{Prob Space} \longleftrightarrow \{ S_0, S_1, S_2, \ldots \}$$

$$(X_0, X_1, \ldots)$$

$S_0 = 1, 2$

$S_1 = 1, 2,$

$S_2 = 1, 2 \ldots$

$(\vec{1}_0, \vec{1}_1, \ldots)$ another, where

$$\text{Prob}\left[\vec{1}_0 \right] = \text{stochastic vector}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{proj}_{\vec{1}}(\ ) + \text{leftover}$$

$$\begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} + \underbrace{\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}}$$

norm

$\frac{2}{3}\sqrt{3}$

$|| \quad ||_2$

$= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$

$$\boxed{A_G = \frac{d}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \varepsilon}$$

$= \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$

$\lambda_1 v_1 v_1^T \qquad \sum_{i \geq 2} \lambda_i \vec{v}_i \vec{v}_i^T$

$= \sqrt{\frac{2(3-2)}{3}}$

parallel to $\vec{1}$

orthogonal to $\vec{1}$

$$A_G \underbrace{Q}_{\text{eigenbasis}} = QD$$

$$Q^{-1} A_G Q = D$$

$$D = \begin{bmatrix} d & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

$$e_I = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{indicator } I,$$

$\Longrightarrow$ If $\vec{v}_1, \dots, \vec{v}_n$ is an ON basis

$$e_I = \underbrace{c_1 \vec{v}_1 + \dots + c_n \vec{v}_n},$$

$$\vec{v}_j \cdot e_I = \vec{v}_j \cdot (\qquad) = c_j \, \vec{v}_j \cdot \vec{v}_j$$

$$e_I = (e_I \cdot \vec{v}_1)\vec{v}_1 + \underbrace{\dots + (e_I \cdot \vec{v}_n)\vec{v}_n}_{\text{other stuff}}$$

$\Longrightarrow$ eigenbasis

$$|I| \cdot \frac{1}{\sqrt{n}} \begin{bmatrix} 1/\sqrt{n} \\ \vdots \\ 1/\sqrt{n} \end{bmatrix}$$

$$= \frac{|I|}{n} \vec{1}$$

"In $\vec{v}_1, \dots, \vec{v}_n$ coords," $\vec{v}_i$ ON eigenbasis of A,

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$D = \begin{bmatrix} d & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

$$A(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)$$

$$= c_1\lambda_1\vec{v}_1 + \dots + c_n\lambda_n\vec{v}_n$$

$$c_1 d\,\vec{v}_1 + \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}$$

if $\lambda_2 = \dots = \lambda_n = 0$,
get 0

if $c_2, \dots, c_n = 0$ get 0

$$A\big(\overbrace{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n}^{\vec{w}}\big)$$

$$= (c_1\lambda_1)\vec{v}_1 + \cdots + (c_n\lambda_n)\vec{v}_n$$

$$A\vec{w} =$$

$$\lambda_1 \,\text{proj}_{\vec{v}_1}\vec{w} + \lambda_2 \,\text{proj}_{\vec{v}_2}\vec{w} + \cdots$$

$$+ \lambda_n \,\text{proj}_{\vec{v}_n}\vec{w}$$

$$= \sum \lambda_i \underbrace{\text{proj}_{v_i}}\vec{w}$$

$$= \Big(\sum (\lambda_i \overbrace{\vec{v}_i)\, v_i \cdot}\Big)\,\vec{w}$$

$$QQ^T = I$$

$$\det(QQ^T) = 1$$

$$= \det(Q)\det(Q^T)$$

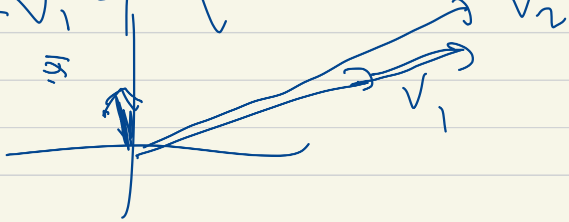
$$= (\det(Q))^2$$

$$\Rightarrow \det Q = 1, -1$$

λ's

$$\vec{v}_2 - \vec{v}_1 \leftrightarrow \begin{pmatrix} -1 \cdot \vec{v}_1 \\ 1 \cdot \vec{v}_2 \end{pmatrix}$$

$\approx$



$$\begin{bmatrix} .1 & 10^{30}/2 \\ 10^{30}/2 & .1 \end{bmatrix}$$

same quad form

$$\begin{bmatrix} .1 & 10^{30} \\ 0 & .1 \end{bmatrix}$$