CPSC 531F, Feb 9

- HW already assigned "due" Feb 26

Reading Break = Feb 15-19

I'll give comments, and you can work more on them & hand them in at the end of course

Upshot! It only matters what you can show me by end of course

Intro: § 4.3 Homework & Supp Notes

later return to expanders, via article by Hoory, Linial, Wigderson

Expander Graphs & Their Applications

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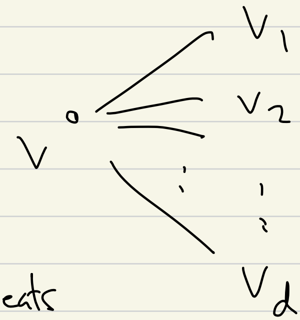
Also use these tools to study Markov chains of various classes

- Reversible MC ⟷ "almost symmetric"
- General " ⟷ eigenvalues tell you a lot, but not everything

Last time:

graph, d-regular:



$v_1, \ldots, v_d$ $\begin{cases} \text{could be repeats} \\ \text{could include } v \text{ itself} \end{cases}$

Thm: If G is d-regular, on n vertices,
if

$\lambda_n(G) \leq \quad \cdots \quad \leq \lambda_2(G) \leq \lambda_1(G)$

are the eigenvalues of A_G, then

$-d \leq \lambda_n(G) \qquad\qquad \lambda_1(G) = d$

- the multiplicity of d as an eigenvalue $=$ # of connected components of G

- $\lambda_n = -d$ iff G is bipartite

[The Expander Mixing Lemma]

$U, W \subset V_G$, $E(U,W)$ = the set of edges from U (i.e. its tail is in U) to W, then

(if ρ small compared to d)

$$\underbrace{\left| E(U,W) - \frac{d}{n}|U|\,|W| \right| \le \underbrace{\rho}_{\text{ideally small}} \sqrt{|U|\cdot|W|}}$$

where

$$\rho = \max_{i \ge 2} |\lambda_i(G)| = \max(\lambda_2, -\lambda_n)$$

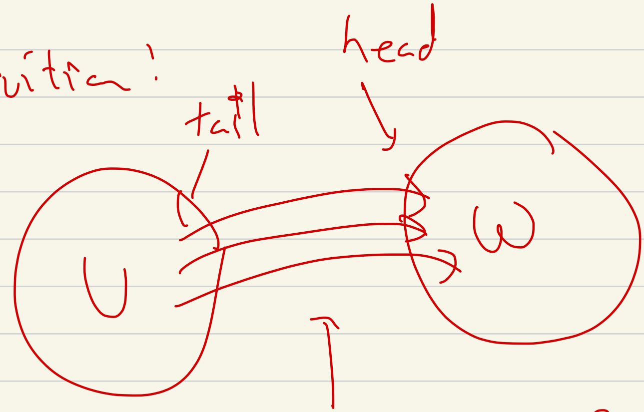
=

Think of d as small compared to n ...

"Expander": d is fixed, $n \to \infty$

We'll learn more precise estimates. then

Intuition!



$|E(U,W)| = ?$ roughly

Random d-reg graph

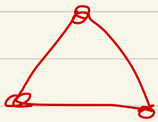
U has $d \cdot |U|$ edges

["expect" $\frac{|W|}{n}$ prob for a single edge to land in W]

total "expected"

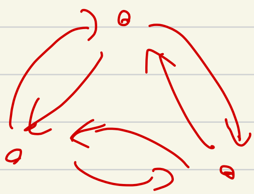
$|E(U,W)|$ roughly $\frac{d}{n} |U| \cdot |W|$

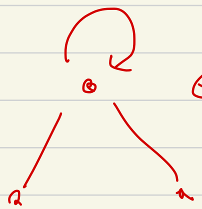
Recall!



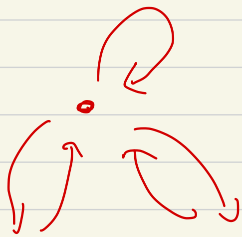
$$A_{\triangle} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

digraph (directed graph)



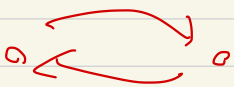


$\leftarrow$ digraph $\rightarrow$



etc etc

convention!



digraph

Last time:

Claim: multiplicity of d as eigenvalue = # connected components.

$A\vec{1} = d\vec{1}$ $\qquad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, d is an eigenvalue

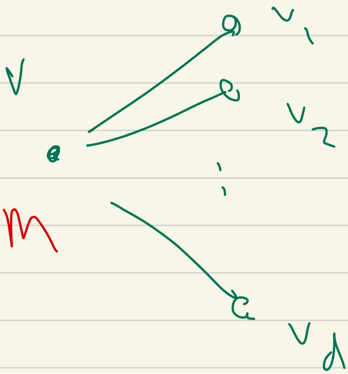
Pf: Say $A\vec{u} = \lambda\vec{u}$

(1) $\lambda \le d$, (2) $\lambda = d \Rightarrow$ something about connectivity

$\vec{u} \neq \vec{0}$, $\vec{u}$ has some pos or neg entry, we can assume $\vec{u}$ has a positive entry (otherwise $\vec{u} \to -\vec{u}$)

Now ~~that~~ $v \in V$ s.t.

$\vec{u}(v)$ is largest possible, $\vec{u}(v) = m$



$\vec{u}(v) = m$

$\vec{u}(v_1) \le m$

$\vec{u}(v_2) \le m$

$\vdots$

$\vec{u}(v_d) \le m$ ...

Want to prove

(1) $\lambda \le d$, (2) If $\lambda = d$, then ...

$$(A_G \vec{u})(v) = \vec{u}(v_1) + \dots + \vec{u}(v_d)$$

$$A_G = \begin{bmatrix} & 1 & 1 & \\ 1 & & 1 & 1 \\ & 1 & 1 & \end{bmatrix}, \text{ rest } 0\text{'s}$$

So $$\underbrace{(A_G\vec{u})(v)}_{\lambda \cdot m} = \sum_{v_i \text{ neighbour of } v} \vec{u}(v_i)$$

$$\lambda m \leq d \cdot m$$

$$\Rightarrow \lambda \leq d$$

What happens if $\lambda = d$?

$$dM \leq \overbrace{dM}$$

but $\sum \vec{u}(v_i)$ equals dM

iff $\vec{u}(v_1) = \dots = \vec{u}(v_d) = M$

If $\quad A_G \vec{u} = d \cdot \vec{u}$

and $\quad \vec{u}(v) = m \quad$ max value of $\vec{u}$

$\lambda = d$ implies

$$\vec{u}(v) = m \quad \begin{cases} \vec{u}(v_1) = m \\ \quad\vdots \qquad\quad = m \\ \quad\vdots \\ \vec{u}(v_d) = m \end{cases}$$

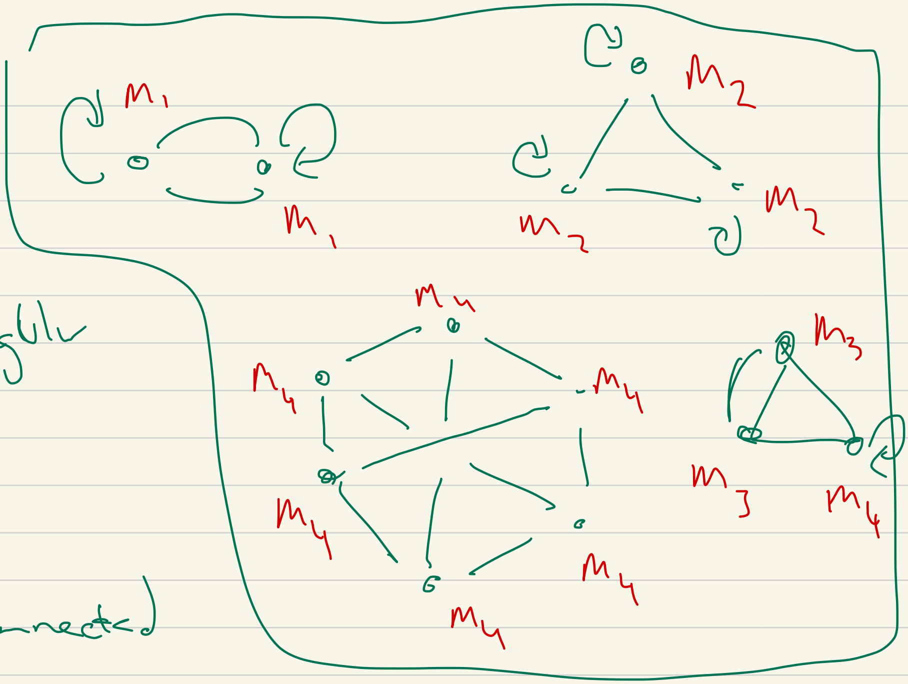
$\Rightarrow$ Similarly

$$m \quad \begin{cases} m \quad \begin{cases} m \\ m \\ m \end{cases} \\ m \\ \vdots \\ m \end{cases} \qquad \begin{matrix} m \\ m \\ m \end{matrix}$$

$\Rightarrow$ any v' connected to v

has $\quad \vec{u}(v') = m$

G :



3-regular

4 connected

components,

maximum principle $\Rightarrow$ $A_G \vec{u} = d\,\vec{u}$,

then $\vec{u}$ is constant on any

connected component.

Converse: If $\vec{u}$ is constant on any

connected component, then $A_G \vec{u} = d\,\vec{u}$.

Claim: Since

$$\{\vec{u} \mid A\vec{u} = d\vec{u}\} =$$

$$\{\vec{u} \mid \vec{u} \text{ is constant on all connected components of } G\}$$

$\Rightarrow$ eigenvalue d has multiplicity $=$ # connected components.

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5 min break

__Def__: The multiplicity of λ as an eigenvalue of A, if __A is diagonalizable__:

"geom multiplicity"

$$= \dim E_\lambda, \quad E_\lambda = \{\vec{u} \mid A\vec{u} = \lambda\vec{u}\}$$

(this equals "algebraic multiplicity" via characteristic polynomial when A is diagonalizable)

E_d has basis:

G

"basis"

Comp 1	1	0	0
Comp 2	0	1	0
Comp 3	0	0	1

	$\vec{r}_1$	$\vec{r}_2$	$\vec{r}_3$
comp 1	1 everywhere	0 everywhere	0 everywhere
comp 2	0 "	1 "	0 "
comp 3	0 "	0 "	1 "

any other $\vec{v}$, could write

$$\vec{v} = \vec{r}_1 c_1 + \vec{r}_2 c_2 + \vec{r}_3 c_3 + \underbrace{\left(\text{orthogonal to } \vec{r}_1, \vec{r}_2, \vec{r}_3\right)}_{\text{leftover part} = \vec{\ell}}$$

$$\vec{\ell} \perp \vec{r}_1, \vec{r}_2, \vec{r}_3 \Rightarrow \sum_{\substack{v \in \text{com} \\ \text{comp}}} \vec{\ell}(v) = 0$$

So if $\vec{l}$ is not $\vec{0}$, then

$\vec{l} \perp \vec{\gamma}_1$: $\vec{\gamma}_1$

compl all 1's,

$\sum \vec{l}(v) = 0$, so $\vec{l}$ has + entry

v in compl

$-$ entry

so $\vec{l}$ is not constant

on $\vec{\gamma}_1$

$\Rightarrow \vec{l} \neq E_d$

Explain in more detail, and prove Exp. Mix..

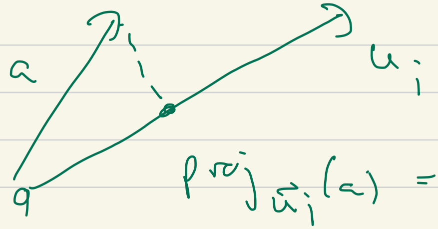
We claim! let $\vec{u}_1, \ldots, \vec{u}_m$ be an orthogonal set of vectors in $\mathbb{R}^n$

Orthogonal: $\vec{u}_i \cdot \vec{u}_i \neq 0$, i.e. $\vec{u}_i \neq 0$

and $\vec{u}_i \cdot \vec{u}_j = 0$.

Then

$$\mathrm{proj}_{\vec{u}_i}(\vec{a}) = \vec{u}_i \left(\frac{\vec{a} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} \right)$$



$\mathrm{proj}_{\vec{u}_i}(a) = \vec{u}_i \cdot C$

If $\vec{u}_i \cdot \vec{u}_i = 1$, $\| u_i \|_{L^2} = 1$

then

$$\text{proj}_{\vec{u}_i}(\vec{a}) = \vec{u}_i \left(\vec{a} \cdot \vec{u}_i \right)$$

$=$

E.g.,

		$\vec{u}_i$	$\vec{u}_i \cdot \vec{u}_i =$
G	comp 1	$\leftarrow$ all 1's	# vert in comp 1
	comp 2	$\leftarrow 0$	
	comp 3	$\leftarrow 0$	

unit vector version of

$$\vec{u}_i = \vec{u}_i \Big/ \sqrt{\#\text{ vert in comp } 1}$$

If $\vec{a} \perp \vec{u}_i$ and

$\vec{u}_i$!

$$\begin{array}{rc} & u_i \\ \begin{matrix} \\ \text{Comp } 1 \leftarrow 1 \leftarrow \\ \\ \\ \\ \text{Comp } 2 \leftarrow 0 \leftarrow \\ \\ \\ \\ \text{Comp } 3 \leftarrow 0 \leftarrow \\ \\ \\ \end{matrix} & \left[\begin{matrix} 1\\1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0 \end{matrix}\right] \end{array}$$

then

$$\vec{a} \cdot \vec{u}_i = \sum_{\substack{\text{sum} \\ v \text{ in Comp } 1}} \vec{a}(v)$$

Remark: If $\vec{a} \in \mathbb{R}^n$, $\left(\begin{array}{c} \text{Graph}, \\ n = \#\text{vertices} \\ A_G \end{array} \right)$

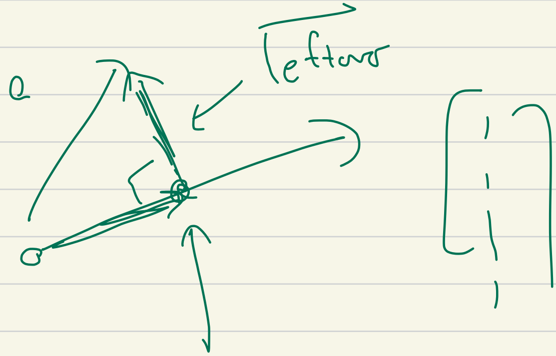
$$\vec{u} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \vec{1}, \quad \vec{u} \cdot \vec{u} = n$$

$$\text{proj}_{\vec{u}}(\vec{a}) = \vec{u} \frac{\vec{a} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

$$\text{proj}_{\vec{1}}(\vec{a}) = \vec{1} \frac{\vec{a} \cdot \vec{1}}{n} = \frac{\vec{1}}{n} \sum_{i=1}^{n} a_i$$

$$= \frac{\sum a_i}{n} \vec{1} = \underset{\text{value}}{\text{avg}}(a_i) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\vec{a} = \left(\text{avg value}(\vec{a}_i)\right)\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} + \overrightarrow{\text{leftover}}$$



$$\text{proj}_{\vec{1}}(a)$$

What is

$$\left|\overrightarrow{\text{leftover}}\right| = ?$$

Answer

$\Rightarrow$ Expander Mixing Lemma

Class Ends

Take any $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ } not linear

$$\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix},$$

$$v, \quad (A-I)v, \quad (A-I)^2 v$$

linearly indep

3.11: $(\sigma_- f)(n) = f(n-1)$

$$(\sigma f)(n) = f(n+1)$$

$$(\sigma_- f)(n) = f(n-1)$$

$$(\sigma^+ f)(n) = f(n+1)$$

$$(\sigma_- f)(n) = f(n-1)$$

$$(\sigma_-) = (\sigma_+)^{-1}$$

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(Jan 26 Band Scan)