Feb 4, 2021 CPSC 531F

- More examples
- d-regular graphs & eigenpairs

simplest

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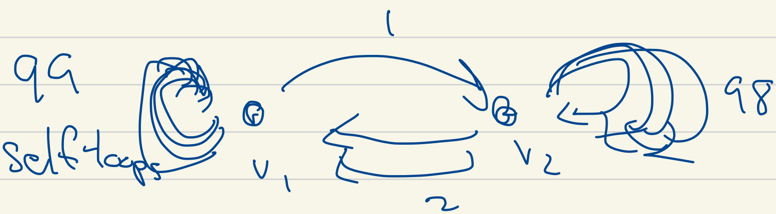
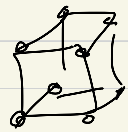
Markov matrix $\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} = \frac{A_G}{100}$



G, $A_G = \begin{bmatrix} 99 & 1 \\ 2 & 98 \end{bmatrix}$

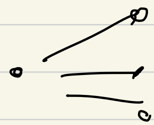
directed graph

Last time: $\mathbb{B}^N$

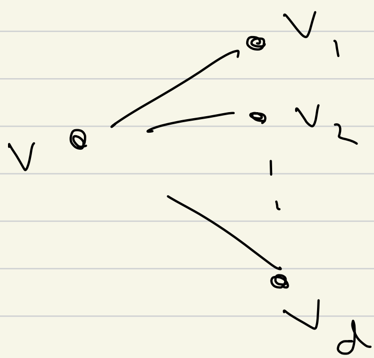


$\mathbb{B}^3$

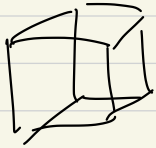
3-regular:

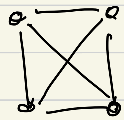


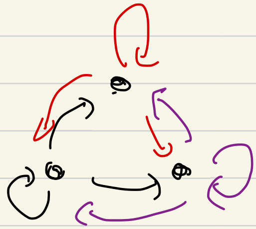
A graph is d-regular if each vertex is incident upon d vertices

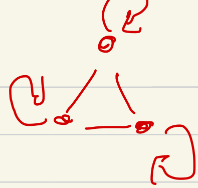


3 regular graphs









$\mathbb{B}^3$ $\qquad$ K_4 $\qquad$ Complete digraph on 3 vertices

"simple graphs"

no self-loops,

no multiple edges

$$A_{\mathbb{B}^3} = \begin{bmatrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

A_{K_4}

8×8

row sums
col sums
$= 3$

$$A_{\mathbb{B}^3} = A_{\mathbb{B}^3}^T$$

A graph, G, means a digraph with

$$A_G = A_G^T$$

Last time

A 

eigenpairs

3 

1

1

1

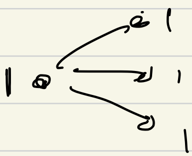
-1

-1

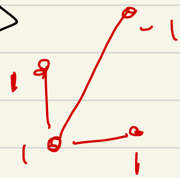
-1

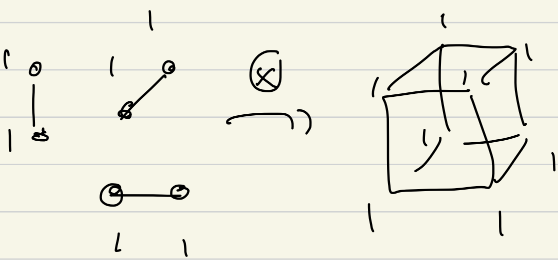
3

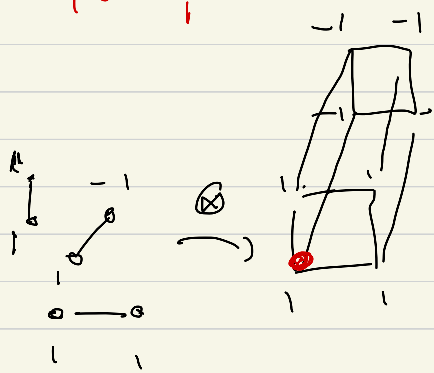
$\lambda = 3$



$\lambda = 1$







See more of later . . .

$$A_{\mathbb{B}^3} = A_{\mathbb{B}^1} \otimes I \otimes I \; +$$

$$I \otimes A_{\mathbb{B}^1} \otimes I \; +$$

$$I \otimes I \otimes A_{\mathbb{B}^1}$$

$\otimes$ "tensor product of matrices"

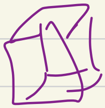
Kronecker " " "

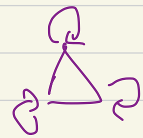
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot 5 & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} 6 \\ \hline \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} 7 & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} 8 \end{array}\right]$$

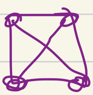
Block

$$= \left[\begin{array}{cc|cc} 5 & 10 & 6 & 12 \\ 15 & 20 & 18 & 24 \\ \hline \text{etc.} & & \text{etc.} & \end{array}\right]$$

eigenvalues, order them

A  $-3 \le -1 \le \ldots \le 1 \le 3$

A  $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $0 \le 0 \le 3$

A  $= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $-1, -1, -1, 3$

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ eigenvalues } 0,0,0,4 \;-\; \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}$$

Thm: Let G be a graph on n vertices, $A_G = A_G^T \in \mathcal{M}_n(\mathbb{R})$, and let eigenvalues of A_G be

$$-d \le \lambda_n \le \quad \dots \le \lambda_2 \le \lambda_1 = d$$

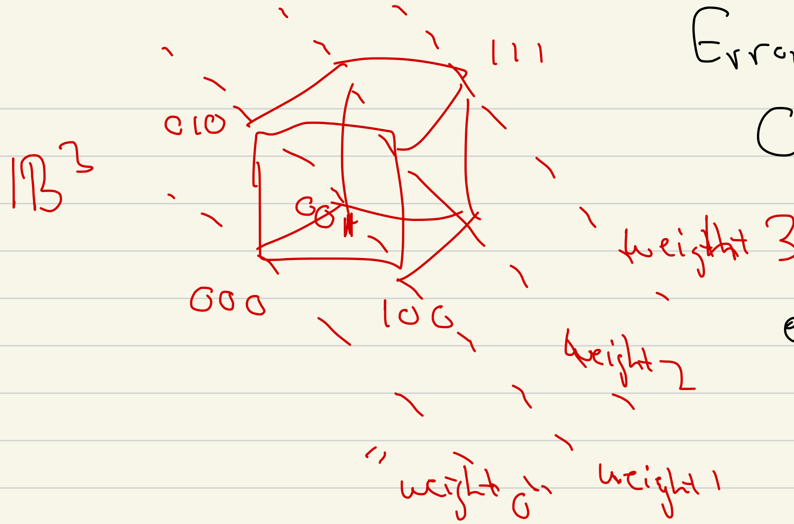
Then if G is d-regular

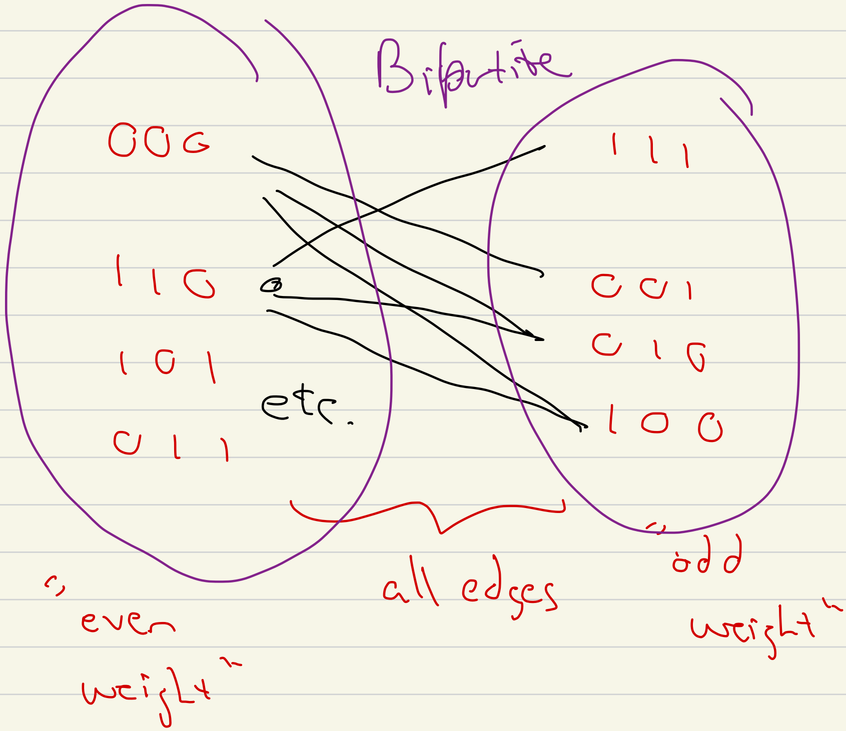
1. $\lambda_1 = d$; moreover the multiplicity of d as an eigenvalue $=$ # connected components of the graph

2. $\lambda_n \ge -d$, and if $\lambda_n = -d$ then G is bipartite

Error Correcting Codes,

esp. Linear Codes





$$\text{weight}\big((x,y,z)\big) = x+y+z$$
$$= \text{sum of components}$$

G is bipartite (any graph) if

$V_G = V_1 \cup V_2, \quad V_1 \cap V_2 = \emptyset$

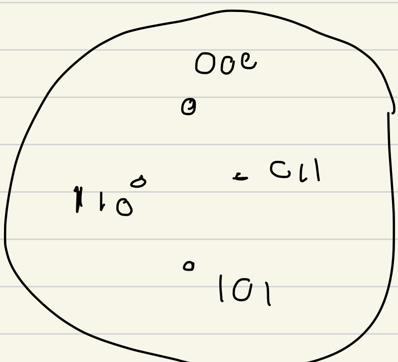
s.t. all edges run from V_1 to V_2.

=

$$-d = \lambda_n \le \dots \le \lambda_2 \le \lambda_1 = d$$

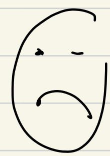
=

G is bipartite $\Longrightarrow$ not a great network



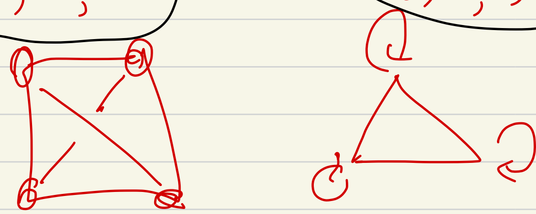
independent set

no one is connected to anyone else in this group



$-1, -1, -1, 3$ $\qquad\qquad$ $0, 0, 3$

Say $G =$



$$G = K_4 \sqcup \text{Compl Dig}_3$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A_G = \begin{array}{c|c} & \begin{array}{cc} 4 & \qquad 3 \end{array} \\ \begin{array}{c} 4 \\ \\ \\ 3 \end{array} & \left[\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & & & \\ 1 & 0 & 1 & 1 & & 0 & \\ 1 & 1 & 0 & 1 & & & \\ 1 & 1 & 1 & 0 & & & \\ \hline & & & & 1 & 1 & 1 \\ & 0 & & & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 \end{array}\right] \end{array}$$

$$\lambda\text{'s}(A_G) = \{-1, -1, -1, 3\} \cup \{0, 0, 3\}$$

no -3 $\qquad$ $-1, -1, -1, 0, 0, 3, 3$

Thm!

$$A = \begin{array}{c|cc} & n_1 & n_2 \\ \hline n_1 & A_1 & 0 \\ n_2 & 0 & A_2 \end{array}$$

"disconnected"
"independent"
blocks

If A is a block matrix as above, then

eigenpairs of A $\Leftrightarrow$ eigenpairs of A_1, A_2

$$\lambda\text{'s} \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 2 & 2 \\ & & & & 2 & 2 \end{bmatrix}$$

blank spaces are 0's

λ's $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $[3]$ $[4]$ $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

Proof of 1 :

$$\lambda_1 = d \qquad \begin{bmatrix} & & & \\ & 1 & 1 & 1 \\ & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = d \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

row sum is d

(more generally: if all row sums of any $A \in M_n(\mathbb{R}, \mathbb{C})$ equal s,

then

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

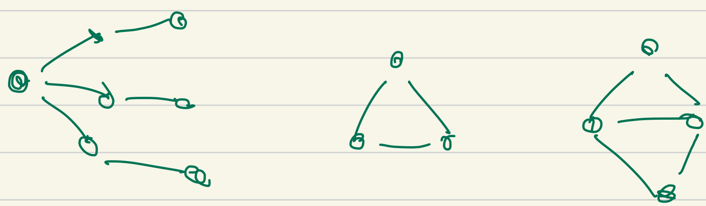
e.g.,
$$\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
)

⋮

Next statement:

multiplicity of d =

\# connected components of G



2 vertices are connected if

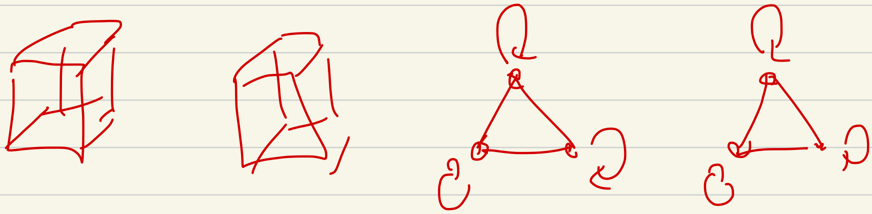
there is a walk from one to the other

a connected component of G

a maximal subset of vertices that

are connected to one another

e.g.,



3-regular, and 4 connected components, and $\lambda = 3$ has multiplicity 4.

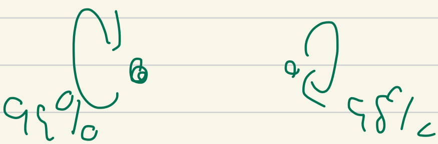
Prove: G d-regular, then the multiplicity of d as an eigenvalue = # connected comps

Proof: ???

Break for 4 minutes

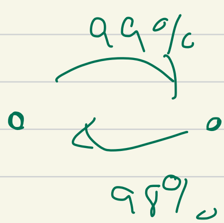
Rem: Aside on MC

$$\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} \qquad \lambda = \underset{\lambda_1}{1}, \underbrace{.97}_{\lambda_2 \simeq 1}$$



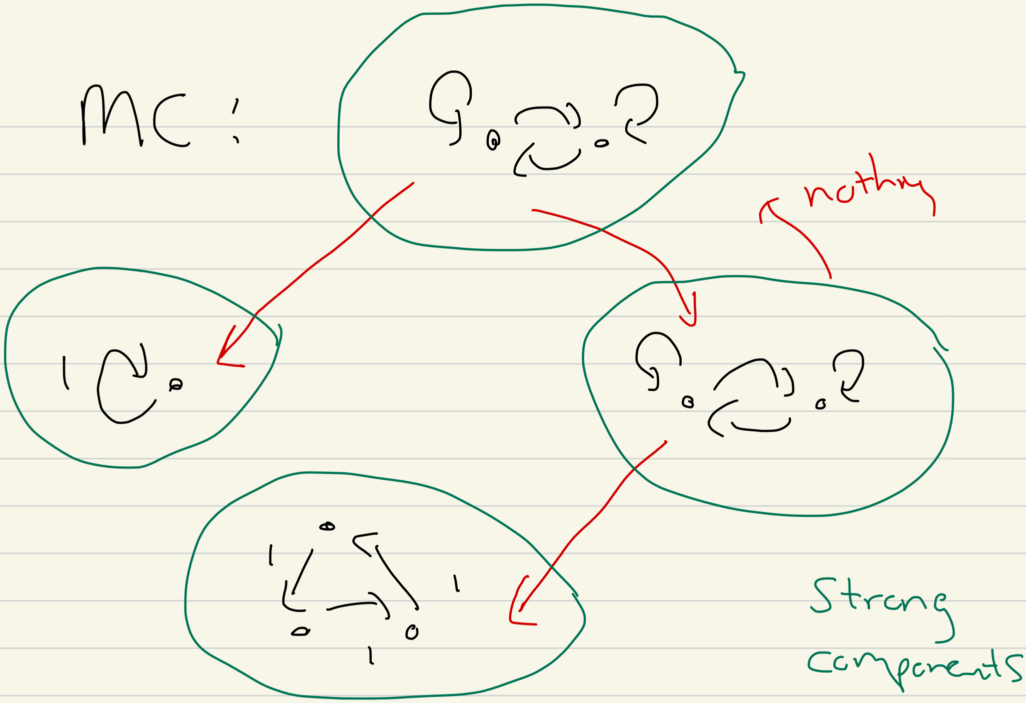
$$\text{Trace} \begin{bmatrix} .99 & \\ & .98 \end{bmatrix} = 1.97 = \text{sum of eigenvalues}$$

$$\begin{bmatrix} .01 & .99 \\ .98 & .02 \end{bmatrix} \qquad \lambda = 1, \underbrace{-.97}_{\lambda_2 \simeq -1}$$



$$\text{Tr} = .03$$

MC:



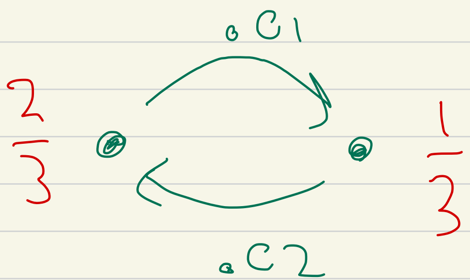
2 states/vertices v_1, v_2 of a Markov chain/associated digraph are strongly connected if there is a walk from v_1 to v_2 and v_2 to v_1

A MC is <u>irreducible</u> if it is strongly connected col sums 1.01, .99 $\neq 1$

MC:

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{\text{stationary distribution}} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

$a, b > 0$, $a + b = 1$ stochastic



$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

Idea of proof

Let $\vec{u}$ be an eigenvector

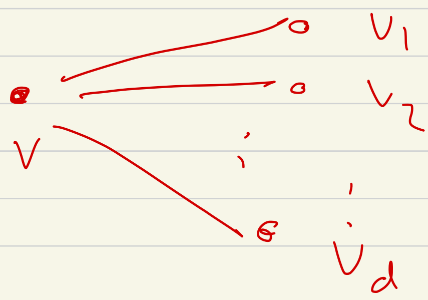
$A\vec{u} = \lambda \vec{u}$, $\quad \vec{u} \neq \vec{0}$

$\vec{u} \in \mathbb{R}^n$

$\vec{u}$ or $-\vec{u}$ has positive comp.

Say $\vec{u}$. Let $\vec{u}$'s max comp be $\vec{u}(v) = M$

$\vec{u}(v) = M$



$\vec{u}(v_1) \leq M$

$\vdots$

$\vec{u}(v_d) \leq M$

Class ends