CPSC 531F, Feb 2, 2002

- Complex ON (orthonormal) bases

$$\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$ ON eigenbasis

for $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

nicer

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \frac{1}{\sqrt{3}},$$ ω = prim. 3rd root of unity

Examples

- Example $\mathbb{B}^n$ eigenpairs

"Theory"

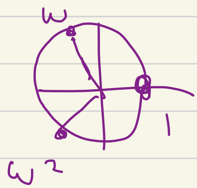
(— Any symmetric matrix has real eigenvalues, ON eigenbasis)

- What do eigenvalues tell us (expanders) vs (clustering)

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad C_3 \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \end{pmatrix} = \zeta \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \end{bmatrix}$$

provided that $\zeta^3 = 1$

$$\omega = e^{2\pi i/3}$$



eigenbasis

$\lambda_1 = 1, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \vec{v}_1$

$\lambda_2 = \omega, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \rightarrow \vec{v}_2 = \frac{1}{\sqrt{3}} \vec{u}_2$

$\lambda_3 = \omega^2, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 = \omega \end{bmatrix}, \quad \vec{v}_3 = \frac{1}{\sqrt{3}} \vec{u}_3$

(Normalize! If A symmetric $\Rightarrow$
$A = \sum \lambda_i \cancel{u}_i \cdot v_i^T$)

Are $\vec{u}_1, \vec{u}_2, \vec{u}_3$ orthonormal?

$$\vec{u}_2 \circ \vec{u}_3 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \circ \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

if you take real dot product

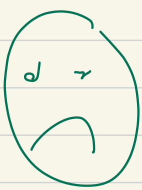
$$1 \cdot 1 + \omega \cdot \omega^2 + \omega^2 \cdot \omega$$

$$1 + 1 + 1 = 3 \;??$$

$$\vec{u}_2 \circ \vec{u}_2 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \circ \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$$

≥ 0, and 0 only for $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

real dot product



$$1 \cdot 1 + \omega \cdot \omega + \omega^2 \cdot \omega^2 = 0$$

$$1 + \omega^2 + \omega = 0$$

What comes to the rescue!

$\vec{a} \in \mathbb{C}^3, \quad \vec{b} \in \mathbb{C}^3$!

$\langle \vec{a}, \vec{b} \rangle$ or $\vec{a} \bullet \vec{b}$

complex dot product

$$= \left\langle \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\rangle$$

definition

$$= \quad a_1 \bar{b}_1 + a_2 \bar{b}_2 + a_3 \bar{b}_3$$

OR $\quad \vec{b}^H \vec{a}$

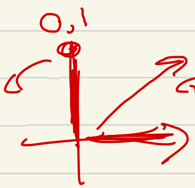
$\vec{b}^H = \overline{\left(\vec{b}^T \right)}$ = transpose, comp. conj of $\vec{b}$

We do know

$z = x + iy$ length $\sqrt{x^2 + y^2}$

$= \sqrt{z \cdot \bar{z}}$

How to rotate by ϑ degrees?



$(\cos\vartheta, \sin\vartheta)$

ϑ

$(1,0) = (\cos 0, \sin 0)$

As a matrix

$$\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Same $z = x + iy$

mult by $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

Note:

$$\langle \vec{u}_2, \vec{u}_3 \rangle = \left\langle \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} \right\rangle$$

$$= 1 \cdot \bar{1} + \omega \cdot \overline{\omega^2} + \omega^2 \cdot \bar{\omega}$$

$$= 1 + \omega \cdot \omega + \omega^2 \cdot \omega^2$$

$$= 1 + \omega^2 + \omega = 0$$

$$\langle \vec{u}_2, \vec{u}_2 \rangle = \left(\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \right)$$

$$= \underbrace{1 \cdot 1}_{1} + \underbrace{\omega \cdot \bar{\omega}}_{1} + \underbrace{\omega^2 \cdot \overline{\omega^2}}_{1} = 3$$

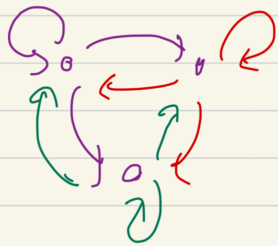
Normalize $\vec{u}_2 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$,

$$\vec{u}_2 / \|\vec{u}_2\| = \vec{u}_2 / \sqrt{\langle \vec{u}_2, \vec{u}_2 \rangle}$$

$$= \vec{u}_2 / \sqrt{3}$$

$$= \begin{bmatrix} 1/\sqrt{3} \\ \omega/\sqrt{3} \\ \omega^2/\sqrt{3} \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longleftrightarrow$ Digraph



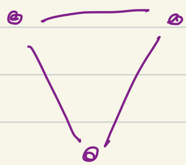
very important to us

Also

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Undirected Graph!

$K_3 \longrightarrow$



complete graph on 3 vertices

If you ever teach lin alg --

$$m = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$m - I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad m - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

If you have ON eigenbasis

for

$$M \text{ — } \lambda_i, \vec{v}_i$$

then for

$$M - aI \text{ — } \lambda_i - a, \vec{v}_i$$

$=$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 3, \ \vec{v}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\lambda_2, \lambda_3 = 0$$

$$||$$

$$\lambda_1 \vec{v}_1 \vec{v}_1^{\,*}$$

Horn & Johnson

$* = T$ in real

$* = H$ in complex case

$$m = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= 3\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$\tilde{m} = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} = A_{K_3} = A_{\triangle}$$

$$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1$$

So

$$\tilde{m} = \underset{\substack{\downarrow\\ 2}}{\lambda_1}\, \vec{v}_1 \vec{v}_1^{\,*} + \underset{\substack{\downarrow\\ -1}}{\lambda_2}\, \underbrace{\vec{v}_2 \vec{v}_2^{\,*}}_{\text{need}} + \underset{\substack{\downarrow\\ -1}}{\lambda_3}\, \underbrace{\vec{v}_3 \vec{v}_3^{\,*}}_{\text{need}}$$

Think of $M \in M_n(\mathbb{R}, \mathbb{C})$ with n distinct eigenvalues

$$M\vec{v}_i = \lambda_i \vec{v}_i \qquad i = 1, 2, \ldots, n$$

($\vec{v}_1, \ldots, \vec{v}_n$ lin indep, hence basis)

$$M \mapsto M - a = M - aI$$

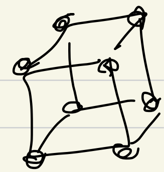
$$(M - aI)\vec{v}_i = M\vec{v}_i - a\vec{v}_i$$

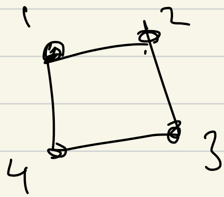
$$= (\lambda - a)\vec{v}_i$$

also $\alpha \in \mathbb{R}, \mathbb{C}$

$$(\alpha M)\vec{v}_i = \alpha(M\vec{v}_i) = \alpha\lambda_i\vec{v}_i$$

$\mathbb{B}^1$ 

$B^3 =$ 

$\mathbb{B}^2$ 

$$A_{\mathbb{B}^1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = C_2$$

eigenbasis

$1, \begin{bmatrix} 1 \\ 1 \end{bmatrix},$

$-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

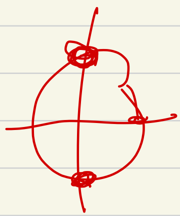
$$A_{\mathbb{B}^2} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = C_4 + C_4^{-1}$$

eigenvectors $\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix}$

$\zeta^4 = 1$

$$(C_4 + C_4^{-1}) \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix} = \overbrace{(\zeta + \quad)}^{\text{real}} \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix}$$

$$\zeta^3 = \zeta^{-1} = \bar{\zeta}$$



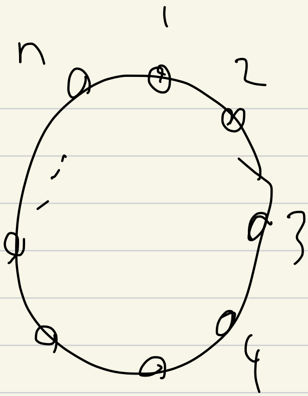
$$\zeta = 1,\ -1,\ i,\ -i$$

$$C_4 \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix} = \zeta \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{pmatrix}$$

$$C_4^2 \left(\cdot \right) = \zeta^2 \left[\quad \right]$$

$$C_4^3 \left(\quad \right) = \zeta^3 \left[\quad \right]$$

$$C_4^{-1} \left(\quad \right) = \zeta^{-1} \left[\quad \right]$$



"Cycle of length n"

$$\mathrm{Adj} = C_n + C_n^{-1}$$

$$\zeta = e^{(2\pi i)\frac{1}{n}\cdot m}$$

$$= \cos\left(\frac{2\pi m}{n}\right) + i\sin\left(\quad\right)$$

$$\bar{\zeta} = \zeta^{-1} = \ldots - i \text{ ---}$$

$$\zeta + \zeta^{-1} = 2\cos\left(\frac{2\pi m}{n}\right)$$

$A_{\square} = A_{\mathbb{R}^2} =$

$\zeta + \bar{\zeta}$, $\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix}$

$\zeta = 1$, 2

$\zeta = i$, $i + \bar{i} = 0 \leftarrow \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$

$\zeta = -i$, $= 0$

$\zeta = -1$, $-2 \leftarrow \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

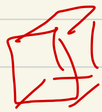
<u>Thm</u>: A symmetric

(1) A has real eigenvalues

(2) " (real) ON eigenbasis

eigenvalues of A  $= -1, 1$

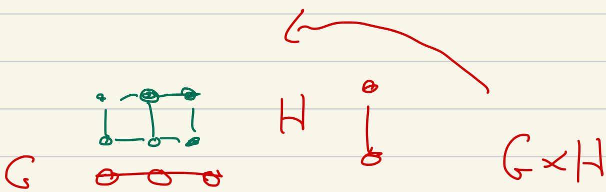
" " A  $= -2, 0, 0, 2$

" " A  $= -3, -1, 1, 3$

mult 3

" " $A_{\mathbb{B}^n} = ?$

$A_{G \times H}$ from A_G and A_H ?



Break for 3 minutes

Claim:

$$\{-1, 1\} + \{-1, 1\}$$

$$\{-1-1,\ -1+1,\ 1+1,\ 1-1\}$$
$$-2 \qquad 0 \qquad 2 \qquad 0$$

$$\{-1, 1\} + \{-1, 1\} + \{-1, -1\}$$

one from each, add

$$1+1+1 = 3$$
$$-1-1-1 = -3$$

$$\left.\begin{array}{l} 1+1-1 \\ 1-1+1 \\ -1+1+1 \end{array}\right\} 1$$, three -1's

Claim! If G, H graphs, have eigenbases (ON eigenbases)

$$A_G \vec{v}_i = \lambda_i \vec{v}_i, \quad A_H \vec{w}_j = \nu_j \vec{w}_j$$

$i = 1, \ldots, n \qquad j = 1, \ldots, m$

$A_{G \times H}$: for all $i \in [n] = \{1, \ldots, n\}$

$j \in [m] = \{1, \ldots, m\}$

$$A_{G \times H}(\vec{v}_i \otimes \vec{w}_j) = (\lambda_i + \nu_j)(\vec{v}_i \otimes \vec{w}_j)$$

e.g.,

$A_{\mathbb{B}^2}$! $A_{\mathbb{B}^1 \times \mathbb{B}^1}$

y

$1, \begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$1, \begin{bmatrix}1\\-1\end{bmatrix}$ $\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$

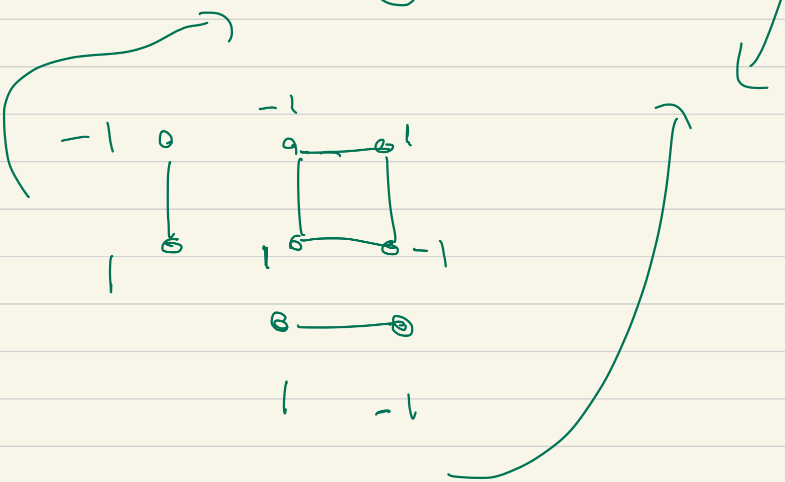
$A_{\mathbb{B}^2}$!

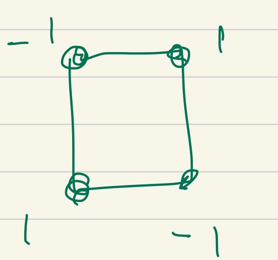
$$1, \begin{bmatrix}1\\1\end{bmatrix} ; \quad -1, \begin{bmatrix}1\\-1\end{bmatrix} \qquad (A_{\mathbb{B}^1})$$

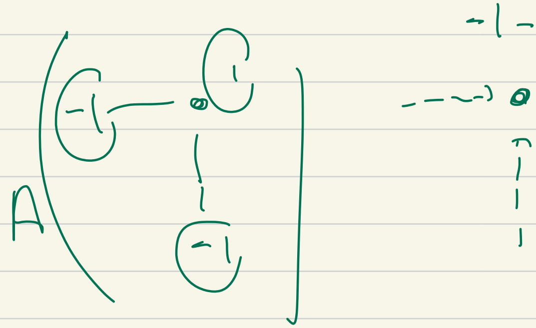
$$1, \begin{pmatrix}1\\1\end{pmatrix} ; \quad -1, \begin{pmatrix}1\\-1\end{pmatrix} \qquad (A_{\mathbb{B}^1})$$

$$|+| := \begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

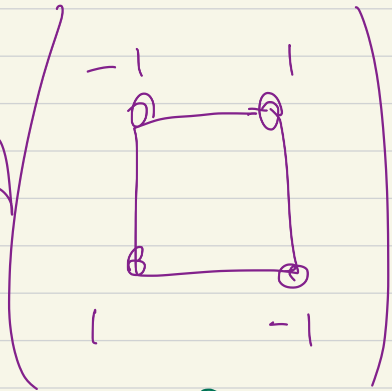
$$-1\ -1 \qquad \begin{bmatrix}1\\-1\end{bmatrix} \otimes \begin{bmatrix}1\\-1\end{bmatrix}$$







$-1 -1 = -2$

A 

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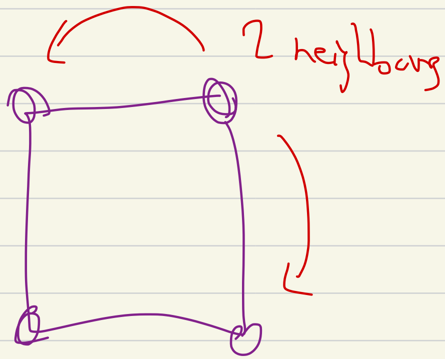
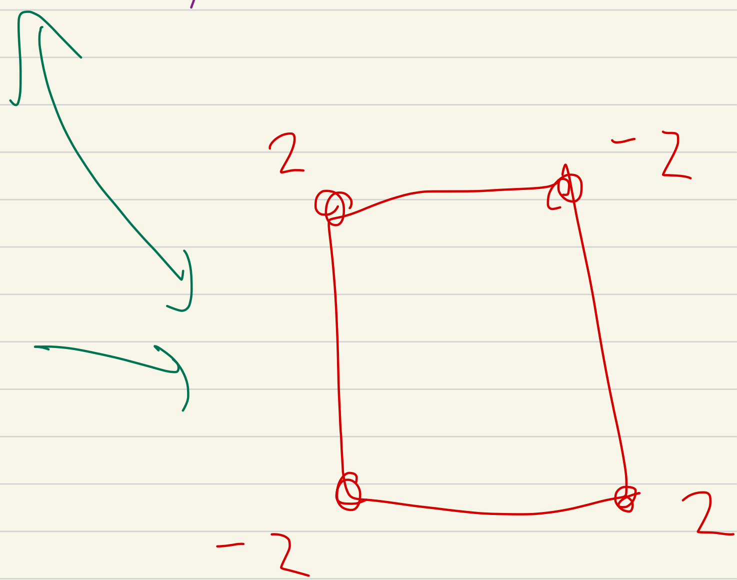
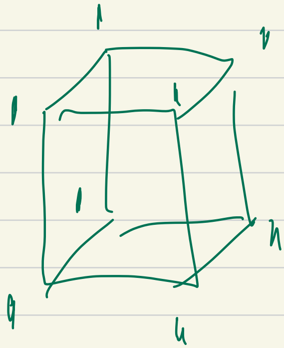


times

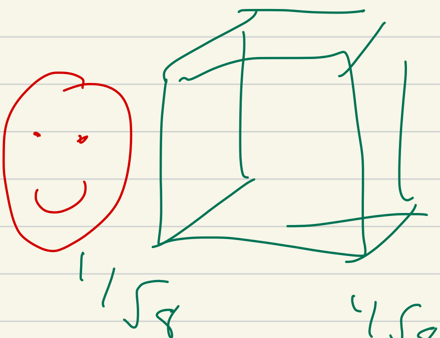
(-2)



$\mathbb{B}^3$



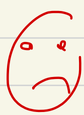
$$\begin{bmatrix}1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\1\end{bmatrix}\otimes\begin{bmatrix}1\\1\end{bmatrix}$$



etc.

$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\otimes\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\otimes\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

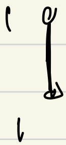
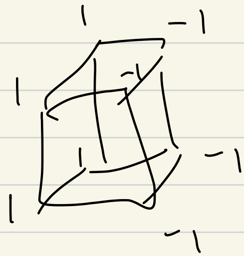
tensor prod

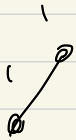


$$\begin{bmatrix}1/\sqrt{8}\\1/\sqrt{8}\\\vdots\end{bmatrix}$$

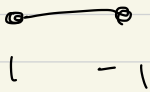
See exercise in notes for

$$A_{G\times H} = A_G\otimes I_H + I_G\otimes A_H$$

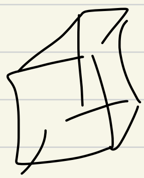
$\lambda = 1$   $\leftarrow \lambda = 1 + 1 + (-1)$



$\lambda = 1$



$\lambda = -1$

eigenbasis of  $\mathbb{B}^n$

Fourier analysis on $\mathbb{B}^n$

 $\lambda = -3$, etc.

and there's another prod:

$$A_{G \underset{\text{prod}}{\text{other}} H} = A_G \otimes A_H$$

$A_{G \times H}$ eigenpairs $<$ A_G, A_H

Class ended
