CPSC 531F, Jan 28, 2021

- Homework:

  Types of Problems

  - More straightforward computations
  - More difficult
    - Computations more abstract
    - "Prove/show" is a bit more formal

e.g.

kernel or null space of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \Rightarrow a+b+c=0$$

2 dim!

free

fixed

ker nullsp $\overbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}^{n} \Big\} n$

amounts to $\begin{bmatrix} \text{all} \\ \text{ones} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0$

$$a_1 + a_2 + \dots + a_n = 0$$

Give a "basis"

$$\underbrace{e_1}_{} - e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

e_1 $\quad$ e_2

$e_1 - e_2,\ e_1 - e_3, \dots, e_1 - e_n$ basis

for ker/nullspace of $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$

parts of a formal proof

<u>Prove</u> that $e_1 - e_2, \dots, e_1 - e_n$ are linearly indep: if

$$\alpha_1 (e_1 - e_2) + \dots + \alpha_{n-1}(e_1 - e_n) = \vec{0}$$

then $\alpha_1 = \dots = \alpha_{n-1} = 0$

Also - More difficult problems

- Problems that require something we won't cover

e.g.

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{eigenvalues: } 1, \omega, \omega^2$$

where $\omega^3 = 1$ but $\omega \neq 1$,

i.e. $\omega = e^{2\pi i/3}, \left(e^{2\pi i/3}\right)^2$

$\rightsquigarrow$ Group Representations

In Class:

$$J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$ a 2x2 matrix

they do come from somewhere...

=

Fibonacci numbers:

$$\boxed{f_{n+2} = f_{n+1} + f_n}$$ ← this

particular "initial data" conditions $f_0 = 0$, $f_1 = 1$

$f_2 = 1$, $f_3 = 2$, ...

↑ without

General solution to

$$f_{n+2} - f_{n+1} - f_n = 0 \quad (*)$$

You <u>guess</u> a simple solution...

hope: $f_n = r^n$, some $r \in \mathbb{R}$

or $r \in \mathbb{C}$

$$r^{n+2} - r^{n+1} - r^n = 0 \quad (\text{say } r \neq 0)$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{5}}{2}, \quad r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

$\Longrightarrow$ for any $c_1, c_2 \in \mathbb{R}$ / $\mathbb{C}$, $\boxed{c_1 r_1^n + c_2 r_2^n}$

solves $(*)$

And so, general solution to

$$f_{n+2} - f_{n+1} - f_n = 0 \quad \forall n \in \mathbb{Z}$$

$$f_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Particular

$$Fib_0 = \boxed{0} = c_1 (\;)^0 + c_2 (\;)^0$$

$$\boxed{= c_1 + c_2}$$

$$Fib_1 = \boxed{1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right)}$$

$$\implies$$

$$Fib_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Sometimes you get double roots in a recurrence

$$x_{n+2} - 2x_{n+1} + x_n = 0$$

Same trick

$$r^2 - 2r + 1 = 0$$

$$r = 1, 1$$

$$x_n = c_1 1^n + c_2 1^n$$

$$= (c_1 + c_2) 1^n$$

$$= c_1 + c_2$$

$x_0 = 5$, $x_1 = 5$, $x_2 = 2x_1 - x_0 = 5, \ldots$

$x_0 = 0, \; x_1 = 1$

$$x_2 = 2x_1 - x_0 = 2$$

$$x_3 = 2x_2 - x_1 = 2 \cdot 2 - 1 = 3$$

$$\vdots$$

double root 1

$$\rightsquigarrow \; x_n = c_1 1^n + c_2 1^n \cdot n$$

$$= c_1 + c_2 n$$

$$=$$

$$x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n = 0$$

$\hookrightarrow$ gen sol

$$x_n = c_1 + c_2 n + c_3 n^2$$

triple root.

Fib matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$$

$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$\leftrightsquigarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = S \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \\ & \frac{1-\sqrt{5}}{2} \end{bmatrix} S^{-1}$$

$$x_{n+2} - 2x_{n+1} + x_n = 0, \quad x_{n+2} = 2x_{n+1} - x_n$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix}$$

How does $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ arise?

$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ eigenvalues are 1, 1

So either $= S \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} S^{-1}$

$SIS^{-1} = I$

OR

$S \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} S^{-1}$

given Jordan canonical form

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

More generally

recur from $(r-\lambda)^2 = 0$

$$r^2 - 2\lambda r + \lambda^2 = 0$$

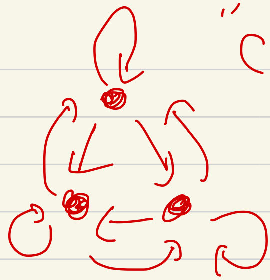
$$\cancel{x}_{n+2} - 2\lambda x_{n+1} + \lambda^2 x_n = 0$$

gen!

$$x_n = c_1 \lambda^n + c_2 \underbrace{n \lambda^{n-1}}_{\text{or } n\lambda^n}$$

Let's get back to

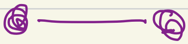
$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \leftarrow$$



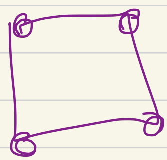
"Complete Directed Graph"

$\cdots$

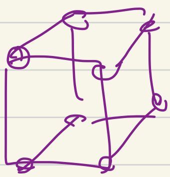
$$\begin{bmatrix}0&1\\1&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{bmatrix}, \cdots$$



$\mathbb{B}$ or $\mathbb{B}^1$,



$\mathbb{B}^2$,



$\mathbb{B}^3$, $\cdots$

Idea!

(1) If $A \in M_n(\overbrace{\mathbb{R}, \mathbb{C}}^{\text{either } \mathbb{R},\ \mathbb{C}})$ that is

symmetric if $A \in M_n(\mathbb{R})$, $A^T = A$

Hermitian --- $(\mathbb{C})$, $A^H =$

complex conjugate $\Big\{ \ \overline{A^T} = A$

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1+3i \\ 1-3i & 5 \end{bmatrix},$$

then

$$A = \sum_{i=1}^{n} \lambda_i \, \vec{v}_i \cdot \vec{v}_i^{\,*}$$

where $\lambda_1, \dots, \lambda_n$ eigenvalues, $\vec{v}_1, \dots, \vec{v}_n$

$\vec{v}_1, \ldots, \vec{v}_n$ are orthonormal eigenbasis

$$A\vec{v}_i = \lambda_i \vec{v}_i, \qquad \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(real) $\vec{v}_i \cdot \vec{v}_j = \vec{v}_i^{\,T} \vec{v}_j$ or usual dot prod

(complex) $\vec{v}_i \cdot \vec{v}_j = \vec{v}_j^{\,H} \vec{v}_i$

Textbook & us: A^* means A^H

real A^* just A^T

Last time

$$\mathbb{B}' \quad \leadsto \quad A_{\mathbb{B}'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

eigenpairs $1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ -1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Normalize:

$$1, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad -1, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

A

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$A = 1 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$+ (-1) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Try the same for

$$\left. A_{\mathbb{R}^2},\ A_{\mathbb{R}^3},\ \ldots \right\} \text{use cartesian products}$$

$$\left. \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{bmatrix},\ \ldots \right\} \text{understand } \mathbb{R} \text{ vs. } \mathbb{C}$$

Popular example:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \lambda\text{'s}: 0, \text{ mult } 2$$

$$3, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ mult } 1$$

Try diagonalizing ---

find O.N. eigenbasis

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0}$$

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \,\middle|\, a+b+c=0 \right\}$$ ⌒ find O.N. basis ...

5-min break

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 3$ $\quad \vec{v}_1$ (normalize)

$$= \vec{v}_1 / \|\vec{v}_1\|_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$\vec{v}_2, \vec{v}_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vec{v}_{2,3} = \vec{0}$$

up to sign

$$\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} Q = Q \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

$$\vec{v}_1$$

Q is orthogonal if (1) $QQ^T = I$

(2) Q's colums are O.N.

(3) " row are O.N.

(4) $\boxed{\vec{u} \cdot \vec{w} = (Q\vec{u}) \cdot (Q\vec{w})}$

why $(Q\vec{u})^T(Q\vec{w}) = \vec{u}^T Q^T Q \vec{w}$

$= \vec{u}^T \vec{w}$

(5) Q preserves $\circ$ product

(6) " " lengths, angles

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 1+\omega+\omega^2 \\ 1+\omega+\omega^2 \\ 1+\omega+\omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\omega^3 = 1$, $\omega \neq 1$, $\omega = e^{2\pi i/3}$

$\omega^3 - 1 = 0 \iff (\omega - 1)(\omega^2 + \omega + 1) = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \end{bmatrix} = \zeta \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \end{bmatrix}, \quad \zeta^3 = 1$$

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} = a \cdot I + b\,C_3 + c\,C_3^2$$

etc.

Next time!

- talk about $\mathbb{C}$-vectors, $\mathbb{N}$-vectors

- ~~cate~~ cartesian products

Class ends

1st prob set – over 1st 4 weeks

due 2 weeks after 4 weeks

= 6 weeks