CPSC 531 F, Jan 26, 2021

Homework + Supplement Notes – document,

supplemental notes
- Give extra examples
- " material not in references
- Outline " not central to CPSC 531
- order will not always correspond to order of topics in class
- some topics we won't cover this year.
- constantly revise throughout term

Yesterday's §4 ⟶ Appendex

Today's §4 ⟶ Examples from Graph Thy, SVD (singular value decomposition), Markov chains

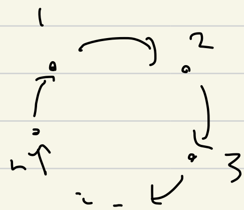
Emphasis on examples (homework).

Examples of matrices, graphs, Markov chains, ...

Last time:

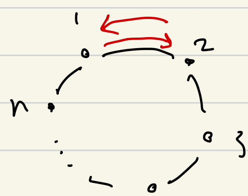
$$C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \vdots & & \\ 0 & & & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$\longleftrightarrow$ (Directed) Cycle of length n



Versus undirected cycle

$$A_{\text{undirected cycle}} = C_n + C_n^{-1}$$



C_n: eigenpairs = eigenvalues/vectors

for any ζ with $\zeta^n = 1$

$$C_n \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} = \begin{bmatrix} \zeta \\ \zeta^2 \\ \vdots \\ [illegible] \end{bmatrix} = \zeta \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \end{bmatrix}$$

we have

"eigenpairs for C_n" : ζ, $\begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$

since $\zeta = e^{2\pi i \frac{m}{n}}$, $m = 0, 1, \ldots, n-1$

this gives n distinct eigenvalues,

$\Rightarrow$ have an eigenvector for each

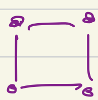
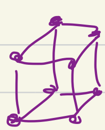
so we have "eigenbasis"

"a basis of eigenpairs"

$A \in M_n(\mathbb{R}, \mathbb{C})$, $\quad A\vec{v}_i = \lambda_i \vec{v}_i$

$\vec{v}_1, \ldots, \vec{v}_n$ basis for $\mathbb{R}^n$ $(\mathbb{C}^n)$, $\quad \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (or $\mathbb{C}$)

Today start: $\mathbb{B}^n$ - boolean hypercube of n dimensions

$\mathbb{B}^1$  , $\mathbb{B}^2$  , $\mathbb{B}^3$ 

A boolean function on n-variables:

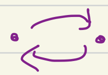
function $\{0,1\}^n \longrightarrow \{0,1\}$

$\{F, T\}^n \longrightarrow \{F, T\}$ $\quad$ F = false, T = true

Studying $\mathbb{B}^n$ can help understand Boolean functions

$\mathbb{B}^1$ $\quad$ 0 •——• 1 $\qquad A_{\mathbb{B}^1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = C_2$

C_2: $\zeta^2 = 1$, $\quad \zeta = 1, -1$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

really 1x1 matrices

notice $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are orthogonal!

for $\vec{u}, \vec{v} \in \mathbb{R}^n$ $\quad \vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$ "=" $\vec{v}^T \vec{u}$

$= \vec{u}^T \vec{v}$

$$\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix} = (1)(1)+(1)(-1) = 0$$

$$\text{“}=\text{”} \quad \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\end{bmatrix}$$

Orthonormal basis:

$$\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$$

$$\text{s.t.} \quad \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Rem: $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \xrightarrow{\text{ON}} \begin{bmatrix}1\\1\end{bmatrix}/\sqrt{2}$

$$\begin{bmatrix}1\\-1\end{bmatrix}/\sqrt{2}$$

become <u>orthonormal</u>

Orthogonal basis $\vec{v}_1, \ldots, \vec{v}_n$ s.t.

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} \text{non-zero} & i = j \\ 0 & i \neq j \end{cases}$$

If $\vec{v}_1, \ldots, \vec{v}_n$ are orthonormal,

then $\vec{u} \in \mathbb{R}^n$,

(*) $$\boxed{\vec{u} = (\vec{u} \cdot \vec{v}_1)\vec{v}_1 + \ldots + (\vec{u} \cdot \vec{v}_n)\vec{v}_n}$$

An orthogonal matrix is a matrix, $Q \in M_n(\mathbb{R})$ s.t. any of the following hold:

(1) Q^{-1} exists and equals Q^T

(2) $QQ^T = I$ (3) $Q^TQ = I$

(4) the columns of Q are orthonormal

(5) " rows " " " "

(6) ($\ast$) holds for all $\vec{u} \in \mathbb{R}^n$

e.g.,

$$C_2 = A_{B'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

ON eigenbasis! $\quad 1, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$-1, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$A_{B'} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 1 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A_{B'} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = (-1) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$A_{B'} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{Q} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underbrace{Q^{-1}}_{Q^T} A Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Fundamental Way to Write a Symmetric Matrix!

- If $A\vec{v}_i = \lambda_i \vec{v}_i$, $\vec{v}_1, \dots, \vec{v}_n$ ON, $\lambda_1, \dots, \lambda_n$ real

then

$$\boxed{A = \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T \quad \text{EXERCISE}}$$

"Spectral thm"

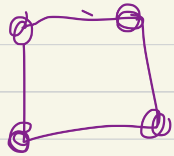
claim! here

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (1)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$+ (-1)\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(check) $= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} (-1)$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

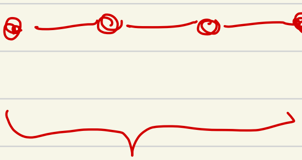
$A_{\mathbb{B}^2}$ =  !

Cartesian product

$= A_{\mathbb{B}^1} \times A_{\mathbb{B}^1}$

$A_{\mathbb{B}^n} = A_{\mathbb{B}^1} \times \cdots \times A_{\mathbb{B}^n}$

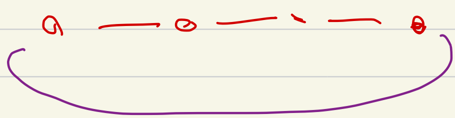
More generally, let's define digraphs/graphs "cartesian product"



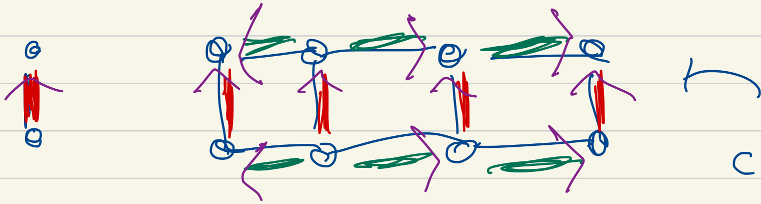
1-dim grid 4 points

note

cycle length 4



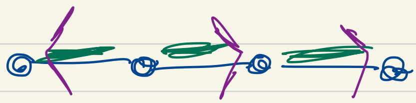
grid
2 pts



(if directed)

cartesian product is 2-dim 4×2 grid

grid
4 points



<u>Definition</u>: Let G, H be two directed graphs

$$G = (V_G, E_G, t_G, h_G), \quad H = (V_H, E_H, t_H, h_H)$$

cartesian product $G \times H$:

(1) $V_{G\times H} = V_G \times V_H = \left\{ (v_1, v_2) \;\middle|\; \begin{array}{l} v_1 \in V_G \\ v_2 \in V_H \end{array} \right\}$

Recall: S, T sets $\quad S \sqcup T$ disjoint union

② $E_{G \times H} = (\quad) \amalg (\quad)$

each edge in G, each vertex of H

$$(E_G \times V_H) \amalg (V_G \times E_H)$$

$$t(e, v) = (t(e), v)$$

$$h(e, v) = (h(e), v)$$

$$t(v, e) = (v, t(e))$$

$$h(v, e) = (v, h(e))$$

Recall! S, T sets, $S \times T = \{(s,t) \mid s \in S, t \in T\}$

Also $S' \amalg T$ = union of S' and T regarded as different sets

often $\boxed{S \times \{0\} \cup T \times \{1\}}$

$(1,2) \amalg (3,4) = (1,2,3',4')$

$(1,2) \amalg (2,3) = (1,2,2',3')$

Aside on disjoint union

$$\begin{tikzcd} 5 \arrow[d, "\|"'] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

product

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\|"'] & \underset{2}{\bullet} \end{tikzcd}$$

$$\begin{tikzcd} (5,1) & (5,2) \\ (1,1) \arrow[from=u, "{(1,1)}"'] \arrow[r, "{(1,1)}"'] & (1,2) \end{tikzcd}$$

the two $(1,1)$ edges are different

Grid graphs

$$0 \xrightarrow{1} 1 \xrightarrow{2} 2 \xrightarrow{3} 3$$

Upshot:

Let G, H be (di)graphs, with eigenbases

$$A_G \vec{v}_1 = \lambda_1 \vec{v}_1, \ldots \quad A_G \vec{v}_n = \lambda_n \vec{v}_n$$

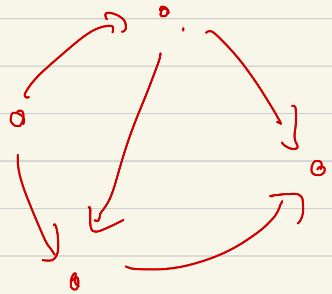
$n = |V_G|$,

$$A_H \vec{u}_1 = \nu_1 \vec{u}_1, \ldots, \quad A_H \vec{u}_m = \nu_m \vec{u}_m$$

then an eigenbasis for $A_{G \times H}$ is . . . [NEXT TIME]

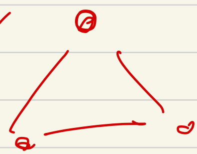
In general!

G directed

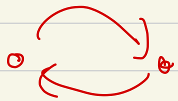


undirected version has

$$A_{\text{“undirected version of } G\text{”}} = A_G + A_G^T$$

$$A\Bigg($$  $$\Bigg) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

we have ON eigenbasis



$$C_2 : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, 1$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, -1$$

perpendicular

not unit vectors

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

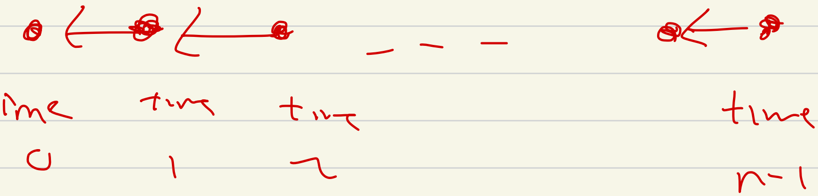
$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Moving Average

grid graph n vertices

directed graph

grid (di)graph



$f: V \to \mathbb{R}$

$$(\text{Mov Avg})(\text{time } t) = \frac{f(t) + f(t-1) + \dots + f(t-6)}{7}$$

nicer!

C_n

(moving avg): $f \to \dfrac{f + C_1 f + \dots + C_6 f}{7}$

nicer $\mathbb{Z}$-grid graph

$$\begin{array}{ccccccccc} \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & & \cdots \\ -1 & & 0 & & 1 & & 2 & & \end{array}$$

$$C_\infty = \begin{array}{c|c} & \begin{array}{ccc} \cdots & 0 & 1 \; \cdots \end{array} \\ \hline \begin{array}{c} \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{array} & \begin{bmatrix} \ddots & \vdots & \vdots & \\ & 0 & 0 & \\ & 1 & 0 & \\ 0\,0\,0 & 0 & 1 & 0\,0\,0 \\ & 0 & 0 & \\ & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

$$\begin{array}{ccccccccc} & \cdots & +1 & -1 & +1 & -1 & & \cdots & \\ \leftarrow & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \rightarrow \end{array}$$

smoothed out

(completely smoothed out) $\Leftrightarrow \ker(I + C + \dots + C^6)$