CPSC 531

- Giving some applications of "spectral" matrix theory, spectral = eigenvalues/vectors, to
  - graph theory: directed graphs from info thy: largest eigenvalue

    Perron-Frobenius

    graphs: symmetric matrices

    Markov matrices: mixing, P^k as $k \to \infty$?

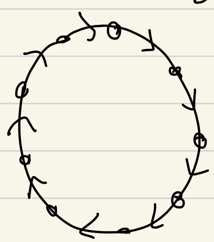
    reversible Markov chains

    simulated annealing

    varying the stationary distribution

Familiarize ourselves with common matrices

Last time:



Directed Cycle on n vertices

$A_{(\ )} = C_n$

$$C_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = [a_{ij}] \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if edge } i \to j \\ 0 & \text{otherwise} \end{cases}$$

$$\underbrace{[1\ 0\ 0\ 0]}_{e_1^T} C_4 = \underbrace{[0\ 1\ 0\ 0]}_{e_2^T}$$

vectors $\vec{v}$: col vect

row vector : $\vec{v}^T$

Goal! - Symmetric matrices have real eigenvalues with orthonormal eigenbasis

- Any positive, square matrix (or non-negative, irreducible) has a largest eigenvalue λ_1 in abs value,
  - $\lambda_1 > 0$, real
  - eigenvector can be taken to have all positive entries

Markov chain, directed graph

undirected graph

reversible Markov chain

C_4 : Eigenvalue/vectors

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{C_4} \underbrace{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}_{} = \text{scalar} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

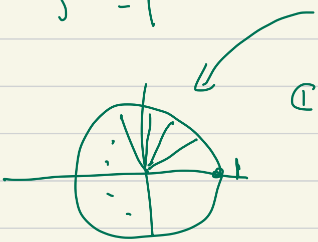
eigenvector (right eigenvector)

Also: left eigenvector

$$[\qquad] C_4 = \text{scalar} [\qquad]$$

Eigenvector! for C_n, any $n \in \mathbb{N} = \{1,2,3,\dots\}$

$\zeta^n = 1$ $\qquad \zeta = e^{2\pi i\left(\frac{m}{n}\right)}$ $\qquad$ "n-th roots of unity"



$$C_n \begin{bmatrix}1\\ \zeta\\ \zeta^2\\ \vdots\\ \zeta^{n-1}\end{bmatrix} = \begin{bmatrix}\zeta\\ \zeta^2\\ \zeta^3\\ \vdots\\ \zeta^n=1\end{bmatrix} = \zeta \begin{bmatrix}1\\ \zeta\\ \vdots\\ \zeta^{n-1}\end{bmatrix}$$

each ζ have an eigenvector, eigenvalue is ζ.

<u>Thm</u>! If $A \in M_n(\mathbb{C})$, and

(1) $p_A(t) = \det(t \cdot I - A)$ has n distinct roots $\lambda_1, \dots, \lambda_n$

(2) there are $\lambda_1, \dots, \lambda_n$ distinct s.t. $A\vec{v}_n = \lambda_n \vec{v}_n$ for some $\vec{v}_j \neq 0$, $j = 1, \dots, n$

then $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent

(lin ind! there are no nontrivial linear relations between the vectors; any n linearly independent vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ are a <u>basis</u>: each vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) can be written uniquely as $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_n\vec{v}_n$

$\alpha_1, \dots, \alpha_n$ are scalars (i.e. $\mathbb{R}$, $\mathbb{C}$).

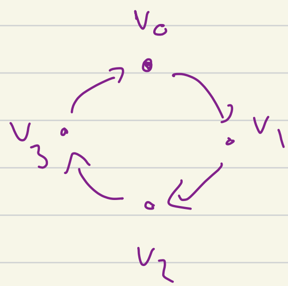
C_4: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$

$\zeta = 1$ $\quad \zeta = i$ $\quad \zeta = -1$ $\quad \zeta = -i$

C_4 is an example of a circulant matrix

(1) $b_0 I + b_1 C_4 + b_2 C_4^2 + b_3 C_4^3$

(2) $\begin{bmatrix} b_0 & b_1 & b_2 & b_3 \\ b_3 & b_0 & b_1 & b_2 \\ b_2 & b_3 & b_0 & b_1 \\ b_1 & b_2 & b_3 & b_0 \end{bmatrix}$



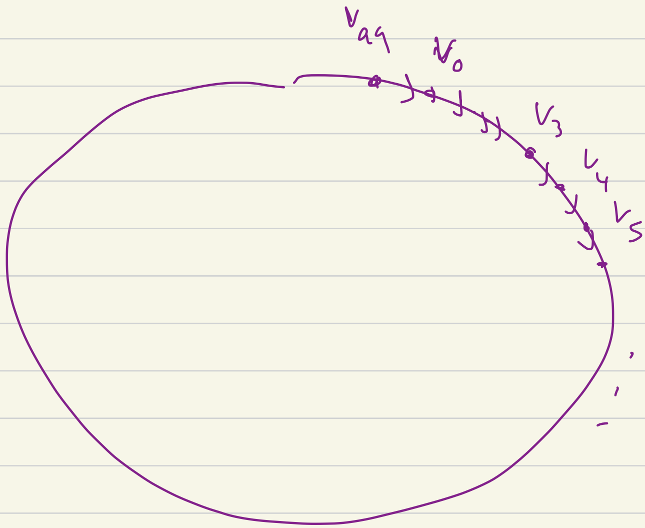
Moving average

$\quad f(v_6) \quad f(v_7)$

$\dots \; \underset{v_3}{\circ} \to \underset{v_4}{\circ} \to \underset{v_5}{\circ} \to \underset{v_6}{\circ} \to \underset{v_7}{\circ} \to \dots$

moving last 3 $\begin{Bmatrix} \text{days} \\ \text{hours} \\ \text{seconds} \end{Bmatrix}$:= at v_7: $\dfrac{f(v_5)+f(v_6)+f(v_7)}{3}$

moving avg on n large



C_n has eigenvalues

$\zeta, \ \zeta^n = 1$

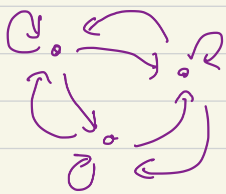
eigenvectors

$$\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} = E_n$$

"complete directed graph adjacency matrix"

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$



turns out: n eigenvalue multiplicity 1

0 " multiplicity $n-1$

"Algebraic multiplicity"

Multiplicity of eig λ of A :

mult. of λ as a root of $p_A(t)$

polynomial

$$p_A(t) = \det(tI - A) \qquad A \in M_n(\mathbb{R})$$

$$= t^n + r_1(A)\, t^{n-1} + r_2(A)\, t^{n-2} + \cdots$$

$$\begin{bmatrix} t & & & & \\ & t & & & \\ & & t & & \\ & & & t & \\ & & & & \ddots \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

$$\det \begin{bmatrix} (t-a_{11}) & -a_{12} & -a_{13} & \cdots \\ -a_{21} & (t-a_{22}) & -a_{23} & \cdots \\ -a_{31} & -a_{32} & (t-a_{33}) & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

$$\det(B) = \sum_{\sigma \in \mathrm{Perm}(n)} b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$

$(\pm 1) \times$

$$\det \begin{bmatrix} (t-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (t-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (t-a_{33}) \end{bmatrix}$$

sign(σ)

$= (t-a_{11})(t-a_{22})(t-a_{33}) \pm$ other prod of entries

$= t^3 + t^2(-a_{11}-a_{22}-a_{33}) +$

$+$ terms of deg 1 or 0

$$t^n + t^{n-1}\underbrace{(-a_{11}-a_{22}-\dots-a_{nn})}_{-\operatorname{Trace}(A)} + \dots + t^1(\ ) + t^0(\ )$$

$$\det \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

$= (-1)^n \det(A)$

Break 10:30 — 10:34

Matrices that aren't diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad p_A(t) = (t-2)^2$$

eigs $\lambda = 2$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}}_{\begin{bmatrix} 2a+b \\ 2b \end{bmatrix}} = \begin{bmatrix} 2a \\ 2b \end{bmatrix} \qquad 2a = 2a+b \implies$$

$b = 0$,

$$\text{eigenspace}_{\lambda=2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left(\begin{matrix}\text{eigenspace of } \lambda \\ \text{w.r.t } A\end{matrix}\right) := \{ \vec{v} : A\vec{v} = \lambda \vec{v} \}$$

$$= \ker(A - \lambda I)$$

$$\left(\begin{matrix}\text{generalized eigenspace of } \lambda \\ \text{w.r.t } A\end{matrix}\right) := \bigcup_{m=1,2,\ldots} \ker\left((A - \lambda I)^m\right)$$

Thm: Any $A \in M_n(\mathbb{R}, \mathbb{C})$ is similar to a block diagonal matrix with blocks

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \in M_{k\times k}(\mathbb{R}, \mathbb{C})$$

$=$

E.g.,

Sometimes $A = S \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} S^{-1}$

Sometimes

$$A = S \begin{bmatrix} 2 & 1 & & & & \\ 0 & 2 & & & & \\ & & 3 & & & \\ & & & 7 & 1 & 0 \\ & & & & 7 & 1 \\ & & & & & 7 \end{bmatrix} S^{-1}$$

or

$$S \begin{bmatrix} 3 & 1 & & & & & & \\ & 3 & 1 & & & & & \\ & & 3 & & & & & \\ & & & 7 & 1 & & & \\ & & & & 7 & & & \\ & & & & & 8 & & \\ & & & & & & 8 & \\ & & & & & & & 8 \end{bmatrix} S^{-1}$$

blank spaces = 0's

$$J_1(\lambda) = [\lambda]$$ ← eigenvector

$$J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$ ←

$$J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$ ←

only one eigenvector per block

$$J_3(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_1 \to v_2 \to v_3$$

$$\left(J_3(0)\right)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left(J_3(0)\right)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

They come up for a reason...

$$x_n - 2x_{n-1} + x_{n-2} = 0$$

$$x_n = 2x_{n-1} - x_{n-2}$$

$$\rightsquigarrow \quad \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

"defective matrices"

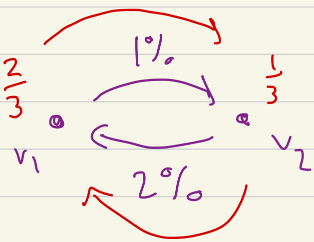
" geometric mult = dim space eigenvectors

$= \dim \ker(A - \lambda I)$

Class ends

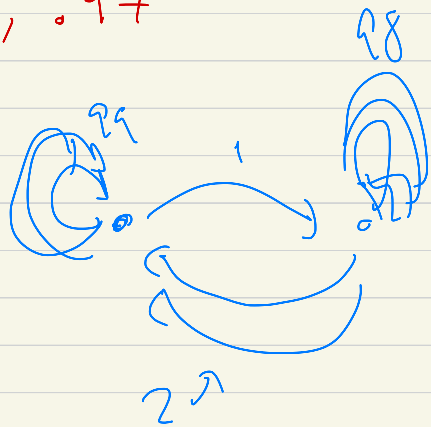
Markov matrix

$$\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} \xrightarrow{\text{large}} \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$



$$\lambda = 1, \ .97$$

balance



$$\begin{bmatrix} 99 & 1 \\ 2 & 98 \end{bmatrix}$$

$$\begin{bmatrix} 99 & 1 \\ 2 & 98 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 100 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 99 & 1 \\ 2 & 98 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \cdot 100$$