CPSC 531, Jan 19, 2021

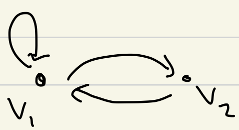
Aiming:
- Perron Frobenius theorem
- Symmetric Matrices have an orthonormal basis (has a version for reversible Markov chains)

- Reviewing motivation
- " eigenvalues/vectors

Motivation: We have digraph (directed graph) or graph, G, want to know: A_G = adjacency matrix:

① What is A_G^k for large k

e.g. Fibonacci graph:



$$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_G \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$A_G \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a \end{bmatrix}$$

$F_0 = 0,\ F_1 = 1,\qquad F_n = F_{n-1} + F_{n-2} \quad n \geq 2$

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ 3, 5, 8, 13, 21, 34, \ldots$

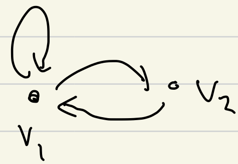
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \quad & \quad \\ & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \quad & \quad \\ & \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ldots$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$



(d,k) - constrained data

Look at $(2,7)$ - constrained data: strings over $\{0,1\}$ s.t. between any two consecutive 1's there are ≥ 2 0's, and ≤ 7 0's

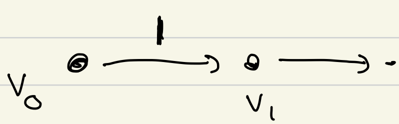
e.g. 000, 1000010010000 1 ...

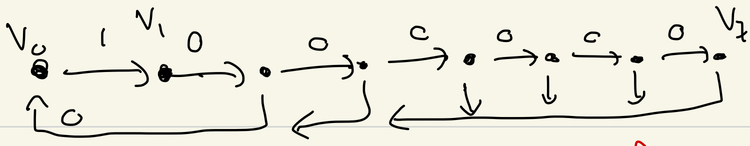
motivation: magnetic storage!

← magnets

1 = flip in polarity

0 = use same polarity





(2,7)-constrained data graph

Question: How many code words of a given length?

$(A_G^k)_{v_0 v_i}$ = # words starting with a 1 ending with $i-1$ zeros, $1 \le i \le 7$

"Shannon capacity" := $\log_2$(largest eigenvalue of)

$\left\{\begin{array}{l}\text{Clustering}\\ \text{Expansion}\end{array}\right\}$ in digraphs or graphs!

Directed graph (digraph), G,

$$A_G = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix} \in M_n(\mathbb{R})$$

sometimes $\mathbb{R}^{n \times n}$

Standard basis: $\mathbb{R}^n$ or $\mathbb{C}^n$

$$e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\\0\\\vdots\\0\end{bmatrix}, \quad \dots$$

$$\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}7 & 8\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 7$$

$$e_i^T A e_j = a_{ij}$$

$I \subset [n] = \{1, \dots, n\}$ then

$$\underbrace{e_I = \sum_{i \in I} e_i}_{\text{indicator vector of } I} = \begin{bmatrix}0\\1\\0\\0\\1\\0\\0\end{bmatrix}$$

1 in locations I

If $I, J \subset [n]$

$$e_I^T A e_J = \sum_{i \in I, j \in J} a_{ij}$$

G digraph

$$e_I^T A_G e_J = \# \text{ edges from } I \text{ to } J$$

Clusters vs expanders: $I \subset [n]$, $I^{comp} = J$

$$e_I^T A_G e_{I^{comp}} = \text{\# edges from } I \text{ to outside } I$$

$$I^{comp} = [n] \setminus I$$

Clusters: $I \subset [n]$ s.t. $e_I^T A_G e_{I^{comp}}$ is "very small"

Expansion: $e_I^T A_G e_{I^{comp}}$ is "what it <u>should</u> be"

Idea of similarity & eigenvalues/vectors:

$A, B \in M_n(\mathbb{R}), M_n(\mathbb{C})$

If

$$A = S^{-1} B S, \text{ then } \overbrace{\cancel{\text{poly}(A)} = S^{-1}\, \cancel{\text{poly}(B)}\, S}^{\text{was unclear}}$$

$p(A)$ $\quad$ $p(B)$

where p is any polynomial

If

$$A = S \begin{bmatrix} d_1 & & 0\text{'s} \\ & \ddots & \\ 0\text{'s} & & d_n \end{bmatrix} S^{-1}$$

$$A^k = S \begin{bmatrix} d_1^k & & 0\text{'s} \\ & \ddots & \\ 0\text{'s} & & d_n^k \end{bmatrix} S^{-1}$$

$$AS = S \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$A \begin{bmatrix} | & & | \\ s_1 & \cdots & s_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ s_1 & \cdots & s_n \\ | & & | \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ As_1 & \cdots & As_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ d_1 s_1 & d_2 s_2 & \cdots & d_n s_n \\ | & | & & | \end{bmatrix}$$

<u>Def</u>! $\lambda \in \mathbb{R}, \mathbb{C}$ is an <u>eigenvalue</u> of A if $\exists$ non-zero $\vec{v}$ s.t. $A\vec{v} = \lambda\vec{v}$,

$\vec{v}$ is an eigenvector (corresponding to λ).

In the above, $\vec{S_i}$ is an eigenvector corresponding to d_i, and

$$\begin{bmatrix} | & & | \\ S_1 & \cdots & S_n \\ | & & | \end{bmatrix}$$ is invertible, i.e. $\vec{S_1}, \ldots, \vec{S_n}$ is a basis for $\mathbb{R}^n$

We say that A is diagonalizable if there are $\lambda_1, \ldots, \lambda_n$ eigenvalues with corr eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ that are linearly independent (i.e. $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $\mathbb{R}^n$ or $\mathbb{C}^n$) (i.e. any vector in $\mathbb{R}^n$ can be written uniquely as $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$ for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ or $\mathbb{C}$).

Examples!

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$:

characteristic poly $A \in M_n(\mathbb{R})$,

$$p_t(A) = \det(I \cdot t - A):$$

$$A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = 0$$

$\vec{v} \neq 0$ $\quad$ I: $n \times n$ identity $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$\iff$

$\ker(A - \lambda I)$ is at least one dimension

nullspace $(A - \lambda I)$

(i.e. not

$$\left.\begin{array}{l} \ker(B) \\ \text{nullspace}(B) \\ \text{(right) nullspace}(B) \end{array}\right\} = \{\vec{v} \in \mathbb{R}^n \mid B\vec{v} = 0\}$$

$\ker(B) \neq \{\vec{0}\} \iff B$ is not invertible $\iff \det(B) = 0$

$A\vec{v} = \lambda \vec{v} \iff [A - \lambda I]\vec{v} = 0$

$n \times n$ identity $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\vec{v} \neq 0$

$\iff$

$\ker(A - \lambda I)$ is at least one dimension

nullspace $(A - \lambda I)$

(i.e. not

$\iff$ $A - \lambda I$ is not invertible

$\iff$ $\det(A - \lambda I) = 0 \iff \det(\lambda I - A) = 0$

$p_A(t) = (I \cdot t - A)$, $p_A(\lambda) = 0 \iff$

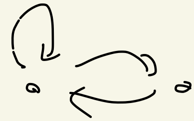
λ is an eigenvalue.

Fib graph

$$\det\left(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right) = 0$$

to find eigenvalues.

10:35 pm – 10:38 pm

Fibonacci graph 

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{pmatrix}$$

$$= (\lambda - 1)\lambda - 1$$

$$= \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{"golden ratio"}$$

$$\lambda_2 \; \frac{1-\sqrt{5}}{2} \quad \text{"its conjugate"}$$

It turns out

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix},$$

i.e. given

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_1 \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{some } a, b$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \ker \left(\lambda_1 I - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = S \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} S^{-1}$$

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \lambda_1/10 & 7\lambda_2 \\ 1/10 & 7 \end{pmatrix}$$

any eigenvector for λ_1

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = S \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} S^{-1}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^n \to \infty \qquad (\lambda_1 \geq 1)$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0 \qquad (|\lambda_2| < 1)$$

as $n \to \infty$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \qquad \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix}$$

$$= (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

[if all ~~rows~~ sums of A are equal, say row sums $= \lambda$]

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{row sum}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$$ row sums of Markov matrix, P, all equal 1

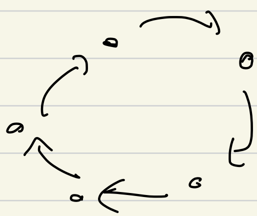
(last class)

$$\mathrm{Tr}(A) = a_{11} + \dots + a_{nn}$$

then $$\lambda_1 + \dots + \lambda_n = \mathrm{Tr}(A).$$

Next time

Cycle matrix!



$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} = C_n$$

I've seen

$$\left(C_n^{-1} + C_n^{-2} + \dots + C_n^{-7}\right)/7$$

= "7 day moving average"

Class ended