CPSC 531F, Jan 14, 2021

Goal Today: Constrained Run-Length Coding

+ Perron-Frobenius eigenvalue

Last time: Matrix notation:

- Stick to ~~textbook~~ by Horn & Johnson

Topic for today: Chapter 1, (HJ) Similarity and Eigenvalues.

$$A \in M_{m,n}(\mathbb{R}) = \mathbb{R}^{m\times n} = \text{set of } m\times n \text{ matrices with real entries}$$

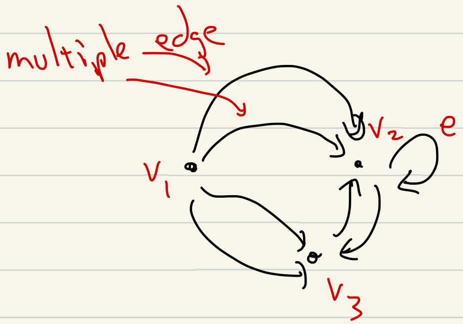
$$M_{n,m}(\mathbb{C}) = \mathbb{C}^{m\times n} = \text{-- -- -- -- --}$$

-- complex entries

Notation

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{m1} & \cdots & & & a_{mn} \end{bmatrix}$$

Directed graph: $G = (\underbrace{V_G}_{\text{vertex set}}, \underbrace{E_G}_{\text{edge set}}, \underbrace{t_G, h_G}_{E_G \to V_G})$ "tails-heads-maps"



self-loop

$t_G(e) = h_G(e) = v_2$

Adjacency matrix of G:

$A_G = [a_{ij}]$, $a_{ij} =$ "# edges from i to j"

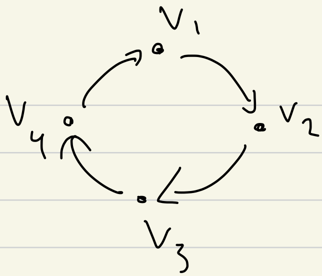
$i \rightrightarrows j$ (i.e. tail $= i$, head $= j$)

$n \times n$ matrix, $n = |V_G|$

Fibonacci graph

$$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

also write $A_G(v_1, v_1) = 1$, $A_G(v_1, v_2) = 1$.



Directed Cycle Length 4

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = C_4$$

$a_{12} = 1$

$a_{24} = 0$

Rem: $[\alpha \ \beta \ \gamma \ \delta] \, C_4 =$

$$[\alpha \ \beta \ \gamma \ \delta] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = [\delta \ \alpha \ \beta \ \gamma]$$

$$[1 \ 0 \ 0 \ 0] \left(\quad '' \quad \right) = [0 \ 1 \ 0 \ 0]$$

Markov chains!

Markov matrix = [HJ]

a (row) stochastic matrix

Stochastic vector: $\vec{x} = (x_1, \ldots, x_n)$ s.t.

$x_i \geq 0, \quad x_1 + \dots + x_n = 1$

"Probability that you are in one of n "states""

Row stochastic matrix: $P = [p_{ij}] \in M_n(\mathbb{R})$

$$= M_{n,n}(\mathbb{R}) = \mathbb{R}^{n \times n}$$

each of whose rows is stochastic.

E.g.

$$\begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}$$

state 1	state 2
in Canada	outside Canada
Favourite TV show: The Expanse	The Mandalorian

Levin Peres

Frog jumping between two lillipads

initial probability distribution

$[80\% \quad 20\%]$

after one month:

$$[.8 \quad .2] \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}$$

$\alpha = .01, \beta = .02$

General 2×2 Markov matrix

$$\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \qquad 1 \geq \alpha, \beta \geq 0$$

After t months (time step),

$$\begin{bmatrix} \text{prob state 1} & \text{prob state 2} \\ \text{time } t & \text{time } t \end{bmatrix} =$$

$$= \begin{bmatrix} \text{---} & \text{---} \\ \text{time } 0 & \text{time } 0 \end{bmatrix} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}^t$$

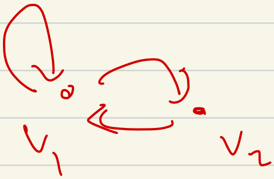
Graph: In a directed graph, a <u>walk</u> of <u>length</u> <u>k</u> in G:

$$v_0, e_1, v_1, e_2 \text{ --- }, e_k, v_k$$

$$(\bullet \longrightarrow \bullet \longrightarrow \bullet \; \text{--} \; \longrightarrow \bullet)$$

s.t. for all $i \in [k] = \{1, 2, \dots, k\}$

$h_G(e_i) = v_i, \quad t_G(e_i) = v_{i-1}$



$(A_G)_{ij}$ = # walks from $i \to j$ length 1

For any $\ell \in \mathbb{N} = \{1, 2, 3, \dots\}$

$(A_G^\ell)_{ij}$ = # walks from $i \to j$ length ℓ

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(1) Given an $A \in M_n(\mathbb{R})$, what does A^ℓ "look like" for large ℓ?

(2) Thm: If A has a positive values, then A has a largest

value (in absolute) λ, and

$$A^\ell \approx \lambda^\ell \left[\text{constant}\right] + o(\lambda^\ell)$$

Perron-Frobenius thm.

Also: if A has non-negative entries, there is a similar result

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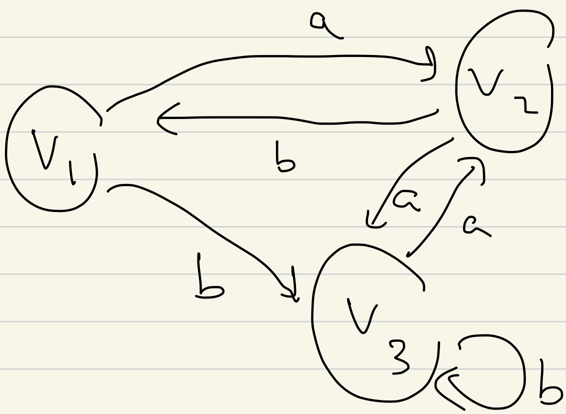
DFA = finite automaton: alphabet Σ (finite set)

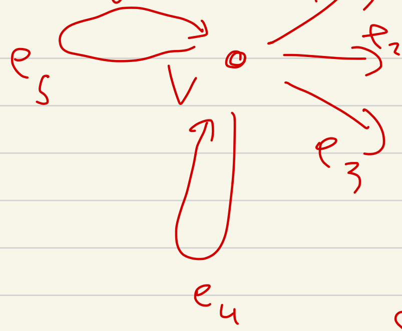
say $\Sigma = a, b$



Directed graph, with labels on the edges.

Here outdegree each vertex is $|\Sigma|$.

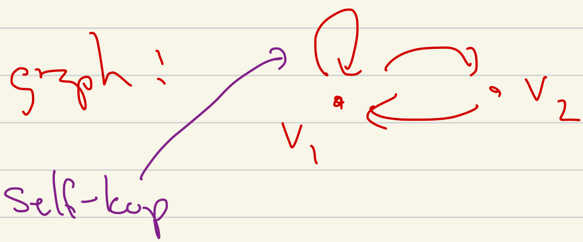
outdegree



the number of edges, e, s.t. $t_G(e) = v$

self-loop: e s.t. $t_G(e) = h_G(e)$

Fibonacci graph:



Ch 1 [HJ]: Similarity and Eigenvalues!

We say $A, B \in M_n(\mathbb{R})$ (or $M_n(\mathbb{C})$) are <u>similar</u> if $B = SAS^{-1}$ for some invertible $S \in M_n(\mathbb{R})$

(equivalently $BS = SAS^{-1}S$

$$BS = SA$$

$$S^{-1}BS = A$$

$$S^{-1}B = AS^{-1}$$

Idea! If

$$A = S^{-1} \underbrace{\begin{bmatrix} d_1 & & 0's \\ & d_2 & \\ 0's & & \ddots\ d_n \end{bmatrix}}_{} S$$

A matrix, D, is "diagonal"

if $D = [d_{ij}]$ and $d_{ij} = 0$ if $i \neq j$

then

$$A^l = S^{-1} \begin{bmatrix} d_1^l & & \\ & d_2^l & \\ & & \ddots\ d_n^l \end{bmatrix} S$$

If so, $d_1, \ldots, d_n$ are "eigenvalues" of A.

Say $A = \underbrace{S^{-1} B S}_{\text{change of coordinates}}$

then

$$A^2 = (S^{-1} B S)(S^{-1} B S)$$

$$= \quad \text{cancel}$$

$$= S^{-1} B^2 S$$

and

$$A^3 = A^2 \cdot A = (S^{-1} B^2 S)(S^{-1} B S)$$

cancel

$$= S^{-1} B^3 S$$

Story ...

$$(AB)^{-1} = B^{-1} A^{-1}$$

(sock on, shoe on)$^{-1}$ = (shoe off, sock off)

Story

$$(S^{-1} B_1 S)(S^{-1} B_2 S) \ldots (S^{-1} B_n S)$$

$$= S^{-1} B_1 \ldots B_n S$$

$S =$ go somewhere $\quad B_i =$ do thing i

$S^{-1} =$ go back

$=$

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^2 = \begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix}$$

If $\quad p(x) = a_0 + x a_1 + \ldots + x^n a_n$

and $\quad A \in M_n(\mathbb{R})$, then

$$p(A) := a_0 I + a_1 A + \ldots + a_n A^n$$

$$A = S^{-1}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} S \quad \leftarrow \text{diagonalization}$$

$$A^3 + 2A + 7 \cdot I = S^{-1}\begin{bmatrix} d_1^3 + 2d_1 + 7 & 0 \\ 0 & d_2^3 + 2d_2 + 7 \end{bmatrix} S$$

$$p(A) = S^{-1}\begin{bmatrix} p(d_1) & 0 \\ 0 & p(d_2) \end{bmatrix} S$$

$$f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

Solve ODE!

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6!} + \cdots$$

Then $e^{At} = \begin{bmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{bmatrix}$

If

$$A = S^{-1} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} S$$

then

$$A^{\ell} = S^{-1} \begin{bmatrix} d_1^{\ell} & 0 & 0 \\ 0 & d_2^{\ell} & 0 \\ 0 & 0 & d_3^{\ell} \end{bmatrix} S$$

if $|d_2|, |d_3| < |d_1|$

then $$A^{\ell} = d_1^{\ell} \, S^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (d_2/d_1)^{\ell} & 0 \\ 0 & 0 & (d_3/d_1)^{\ell} \end{bmatrix} S$$

so as $\ell \to \infty$

$$A^{\ell} = d_1^{\ell} \cancel{(} (1 + o(1)) \, S^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S$$

the smaller $|d_2/d_1|$, $|d_3/d_1|$, the faster

$$\frac{A^\ell}{d_1^\ell} \text{ converges to } S^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S$$

Using $O(\ )$ and $o(\ )$ notation:

$$A^\ell = d_1^\ell \left(1 + o(1) \right) S^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S$$

$$= d_1^\ell \left(I + f(\ell) \right) S^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S$$

$f(\ell) : \mathbb{N} \to M_n(\mathbb{R})$

and $f(\ell) \to 0$!

say each entry of $f(\ell) \to 0$ as $\ell \to \infty$

How do we measure when 2 matrices are "close" $\rightsquigarrow$ matrix norms