CPSC 531F, Jan 12

General Idea, Admin Stuff

Focus of course: "spectral methods" for $\left\{ \begin{array}{l} \text{"complex systems"} \\ \text{graphs} \\ \text{Markov chains} \\ \vdots \end{array} \right.$

Spectral: eigenvalues/vector of large matrices, focus on symmetric matrices...

Idea: You have a large $m \times n$ real/complex matrix, want to infer some "global properties" of the matrix with a few eigenvalues/vectors computation...

- Does a graph have "clusters", or does the graph "expand" well?
- Does a Markov chain "mix quickly"?

  e.g. shuffling cards — how many shuffles of a deck of cards are needed to have a "randomly looking" deck?

  52 cards $\rightsquigarrow$ 52! Possible decks of 52 cards...
- The above are examples where the top few eigenvalues tell you a lot.

- Say you have m points in $\mathbb{R}^n$, $m, n \doteq 10^3, 10^6, \ldots$
Want to represent the m points in a much smaller dimensional space. Arrange points in a matrix, $m \times n$ matrix, use SVD (singular-value decomposition) to get "low rank approximation to $m \times n$ matrix" reducing the m points in $\mathbb{R}^n$ to

" " " $\mathbb{R}^{(\text{much smaller})}$.

- Side remarks: symmetric matrices, unitary " $\}$ common theory of "normal matrices"

unitary matrices $\rightsquigarrow$ Quantum computing

Textbook (reference): - Matrix Analysis, Horn & Johnson

Grading: - Homework divided into 3 sets

3rd homework $\overset{?}{=}$ short survey article

$< 80\%$ unlikely

95% and higher = extremely strong encouragement, and for a recommendation

90% " " = very " " "

85% - 89% = encouraging work in field, you'll have to develop more background
- this is not your main field of research, but you can apply this well

80% - 84% = student has filled expectations of a Ph.D. student in computer science

---

- I'll post these iPad notes
- " " some rough and possibly changing notes
- Classes should be recorded

Start technical material!

- Why are symmetric matrices so prevalent?
- Review the basic theory that we need.

<u>Examples of Symmetric Matrices</u>!

(1) Calculus!

$$f(x) = f(x_0) + (x_0 - x) f'(x_0) + \frac{(x-x_0)^2}{2} f''(x_0)$$

$$+ o(x-x_0)^2$$

in one variable, if

$f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, f twice differentiable.

If $f : \mathbb{R}^n \to \mathbb{R}$, $\vec{x}_0 \in \mathbb{R}^n$

$$f(\vec{x}) = f(\vec{x}_0) + (\vec{x} - \vec{x}_0) \cdot \left[(\nabla f)(x_0)\right] +$$

$$\frac{1}{2} (\vec{x} - \vec{x}_0)^T \underline{\left(D^2 f(x_0)\right)} (\vec{x} - \vec{x}_0) + o\left(|\vec{x} - \vec{x}_0|\right)^2$$

Here $D^2 f$ is the "Hessian"

$D^2 f$ in 2-dim $\qquad \left.\begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix}\right\}$ is symmetric

since

$$f_{x_2x_1} = f_{x_1x_2} \implies$$

Thm! If $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is real, symmetric, 2×2,

then after an "orthonormal change of basis"

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightsquigarrow \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

1. Review what this means
2. There is an $n\times n$ symmetric matrix analog to this theorem.

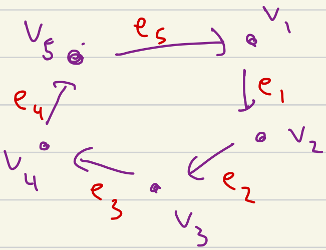
Recommend to look at Horn & Johnson,

Sections 0.0 - 0.6. Chapter 0 = collection of useful facts throughout the textbook

② Graph thy you have symmetric matrices:

Directed graph:

(toddlers running around rooms)



$\bar{V} = \{v_1, \ldots, v_5\}$

$E = \{e_1, \ldots, e_5\}$

formally: ~~a~~ directed graph

$G = (V_G, E_G, t_G, h_G)$

V_G = vertex set of G

E_G = (directed) edge set of G

$t_G : E_G \to V_G$ "tails"

$h_G : E_G \to V_G$ "heads"

$t_G(e_1) = v_1, \quad h_G(e_1) = v_2$

A_G = adjacency matrix : $(A_G)_{ij}$, $i, j \in \bar{V}$

= # edges from $v_i, v_j \in \bar{V}$

i to j

Horn & Johnson: A matrix, entries a_{ij}

$$A \in M_{m,n}(\mathbb{R}) = \mathbb{R}^{m \times n}$$

m rows
n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$

in this notation

$$A \text{ gives map } \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

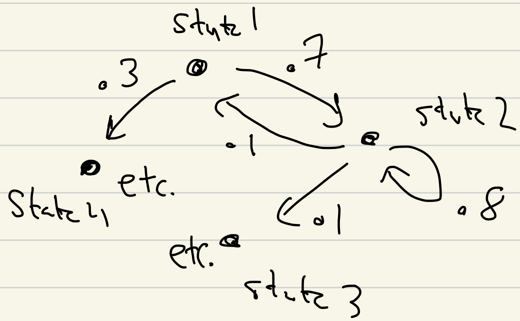
$$v \longmapsto Av$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\text{usual linear alg}} = \begin{bmatrix} a+2b+3c \\ 4a+5b+6c \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{\text{usual way}} = \begin{bmatrix} \alpha+4\beta & 2\alpha+5\beta & 3\alpha+6\beta \end{bmatrix}$$

directed graphs and Markov chains

Markov chain



$\vec{v} \in \mathbb{R}^n$ is <u>stochastic</u> (or a <u>probability vector</u>)

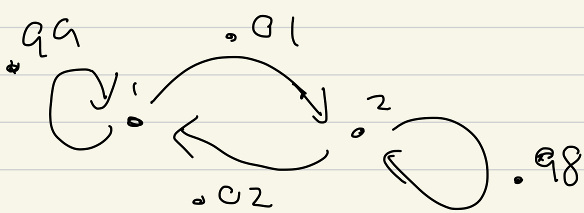
if

$$\vec{v} = (v_1, \dots, v_n) \quad \text{and} \quad v_i \geq 0 \text{ for all } i,$$

$$v_1 + \dots + v_n = 1.$$

A <u>Markov matrix</u>: is an $n \times n$ matrix

whose rows are stochastic vectors.



P_{ij} = prob of moving from state i to j

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} = \begin{bmatrix} \text{new distribution} \end{bmatrix}$$

Class ends

$x_1, x_2 \geq 0$

$x_1 + x_2 = 1$

$$= \begin{bmatrix} .99x_1 + .02x_2, & .01x_1 + .98x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix} = \begin{bmatrix} .99 & .01 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \quad '' \quad = \begin{bmatrix} .02 & .98 \end{bmatrix}$$

x_1 = probability of being in state 1, at some time t

$[x_1 \; x_2]$ time 0, $\quad x_1(0), x_2(0)$

$[\text{new } x_1 \;\; \text{new } x_2]$ time 1, $\quad x_1(1), x_2(1)$

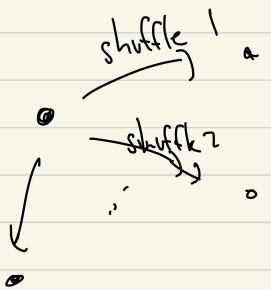
" " " 2, $x_1(2)$, $x_2(2)$

" 3, " 3 3

$$\begin{bmatrix} x_1(t+1) & x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} .99 & .01 \\ .02 & .98 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) & x_1(t) \end{bmatrix} = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix} \begin{bmatrix} & \end{bmatrix}^t$$

52 cards! 52! states



Think of time $t=0$

probability 1

A spades
K "
Q "
J "
10 "
9 "
.
:
2 clubs

People in Vancouver
" " Toronto !

$$\begin{tikzcd} \text{Van} \arrow[loop left, ".99"] \arrow[r, bend right, ".02"'] & \text{Tor} \arrow[l, bend right, ".01"'] \arrow[loop right, ".98"] \end{tikzcd}$$

each month

time $t=0$: 10^6 in Van $\qquad 10^6 \cdot 1.2$ Toronto

$t=1$:

one month later

$$\underbrace{\begin{bmatrix} 10^6 & 10^6 \cdot 1.2 \end{bmatrix}}_{\text{convenient to make this stochastic}} \begin{pmatrix} .99 & .01 \\ .02 & .98 \end{pmatrix} = \begin{bmatrix} \qquad \qquad \end{bmatrix}$$

$$10^6 \begin{bmatrix} 1.0 & 1.2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \frac{1.0}{2.2} & \frac{1.2}{2.2} \end{bmatrix} = \text{stochastic}$$