CPSC 421/501 December 5, 2025

- Today:

Handout in Appendix A

Andre'ev's Function (1987)

$$\underbrace{x_1\ x_2\ \cdots\ x_{n/2}}\ \underbrace{\cdots\ x_{n-1}\ x_n}$$

$\frac{n}{2} = 2^N$ describes function $\{0,1\}^N \to \{0,1\}$ truth table

Divide $n/2$ into N pieces (each piece $n/2N$ bits

Gives formula size $\geq n^{2.5}$

" " " $\geq n^{3-o(1)}$

- What you might have learned in CPSC 506 (continuation of CPSC 421/501) ...

Subbotovskaya 1961

late 1960's, early 1970's

gNTn $\geq n^{2-\varepsilon}$ formula size

different methods than

"random restriction"

related problems

1990s

Andre'ev 1987

lower bound

$n^{2.5}$

$n^{3-\varepsilon}$

1993, 1998

Subbotovskaya's ! $n^{1.5}$ XOR -- $n^{2-\varepsilon}$ AND

Min Circuit Depth vs Size
Min Formula Depth vs Size

4 quantities

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(1) Min Circuit Depth = Min Formula Depth

(2) Min Circuit Size ⇝ P vs. NP

(3) log (Min formula size)

≃ Min formula depth

Highly parallel algorithm ⟺ Circuit

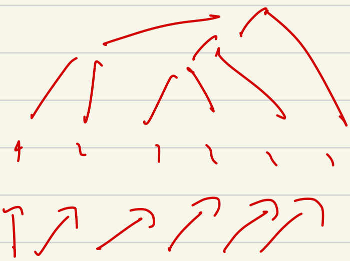
small depth

⇝ NC^1, NC^2, ... vs P vs NP

We know: there are $2^{(2^n)}$ Boolean functions $\{0,1\}^n \to \{0,1\}$

$$\{T,F\}^n \to \{T,F\}$$

just counting # formulas size k



k leaves

$x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n$

tree

each interior vertex: $\wedge, \vee$

$\Rightarrow$ If $k \le 2^n/n$ then

formulas size k $\le \frac{1}{2} 2^{(2^n)}$

$\Rightarrow$

at least 50% of Boolean functions on n variables require formula of size $2^n/n$

($2^n \cdot n$ is sufficient from truth table)

Idea: find an N ($\ll n$) s.t. we can write out all $2^{(2^N)}$ Boolean functions; average min formula required is $(2^N/N)(1/2)$

Andreiev: fix n

$$\boxed{x_1 \ x_2 \ \cdots \ x_{n/2}} \ \boxed{x_{n/2+1} \ \cdots \ x_n}$$

$\downarrow$

1st half:

let N be s.t.

$$n/2 = 2^N$$

so

$$x_1 \ \cdots \ x_{n/2}$$

$$\underbrace{x_1 \ x_2 \ \cdots \ x_{2^N}}$$

Give, $x_i \to T$, $x_i \to F$ every function $\{T,F\}^N \to \{T,F\}$.

2nd half

$$\boxed{x_{n/2+1} \quad x_{n/2+2} \quad \cdots \quad x_n}$$

$n/2$ variables:

divide into N blocks, each

of size $(n/2)/N = n/2N = k$

$$N \left\{ \begin{array}{l} x_{n/2+1} \oplus \cdots \oplus x_{n/2+k} = y_1 \\ x_{n/2+k+1} \oplus \cdots \oplus x_{n/2+2k} = y_2 \\ \vdots \\ \oplus \cdots \oplus \quad x_n = y_N \end{array} \right.$$

$\underbrace{\qquad\qquad}_{k = n/2N}$

$\leftarrow$ 1st collection

$\leftarrow$ 2nd collection

$\leftarrow$ N collection of variables

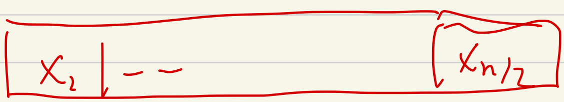
Random restriction of $n - n^\delta$

δ small!

Y_1

$\vdots$ remain XOR each of

Y_N same # of variables



$\leftarrow$ still have enough functions left

restriction

$\Longrightarrow$ Avg formula size for what left at function $Y_1, \dots, Y_N$ is $\gtrsim$ roughly n

$n/2 \doteq 2^N \qquad N \approx \log_2(n/2)$

Andreev:

block of $k \approx \frac{n/2}{N} \approx (n/2)/\log n$

$\oplus$ of k variables size $\approx k^2$

$\qquad\qquad$ " $\approx n^2/(2\log n)^2$

$\Rightarrow$ formula size $\approx n^3$

If you can find function in P

$\qquad\qquad\qquad\qquad\qquad\qquad$ NP

such min formula size $\geq n^{\log(\log(\log n))}$

Final Dec 19

Have a review package on Tuesday
Dec 9

slightly diff 2025

based on everything up
to Subbotovskaya's
method.