CPSC 421/501 Dec 3, 2025

<u>Thm</u> (Subbotovskaya, 1961): Let $f_n$ be a Boolean formula in $x_1, \ldots, x_n$ and $\vee, \wedge, \neg$ for $x_1 \oplus \cdots \oplus x_n$; let

$S_n$ = min size possible of such an $f_n$.

Then $S_n \geq n^{1.5}/\sqrt{8}$.

($S_n \geq n$ immediate)

<u>Improvements</u>:

| | | |
|---|---|---|
| Impagliazzo & Nisan | $n^{1.55}$ | 1993 |
| Paterson & Zwick | $n^{1.63}$ | 1993 |
| Håstad | $n^{2-\varepsilon}$ | 1998 |

See Appendix A of handout.

Rem: If $n = 2^k$, then our size $n^2$ formula for $x_1 \oplus x_2 \oplus \dots \oplus x_n$ has $n$ leaves of each variable ...

$(S_n\,(^{n-1}/_{n}))$

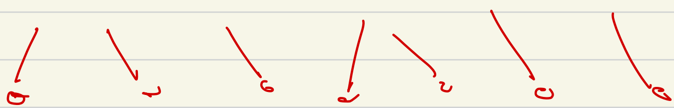
So:

$$S_{n-1} \le S_n\left(1 - \frac{1}{n}\right)$$

very easy, but ...

each variable $x_i, \neg x_i$ can appear close to $\frac{1}{n}$ of the time ...

$x_i$:



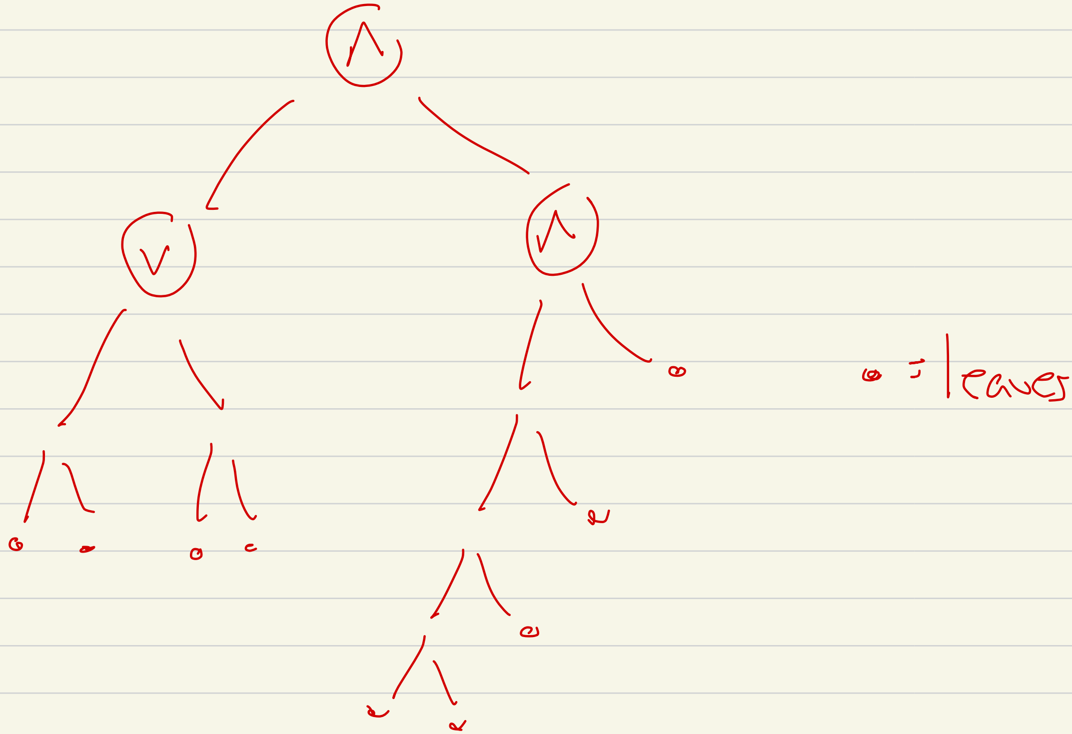
leaves

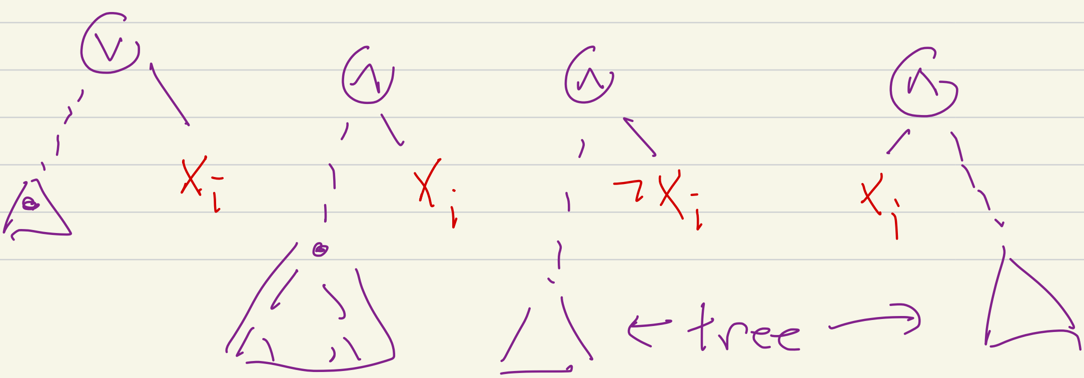
$x_i, \neg x_i$

they can each appear $\approx 1/n$ (# leaves)

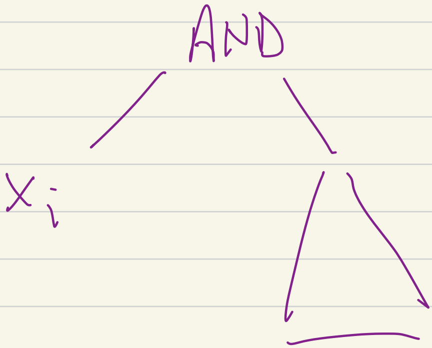
-2.5-

$x_1 \oplus \cdots \oplus x_n$



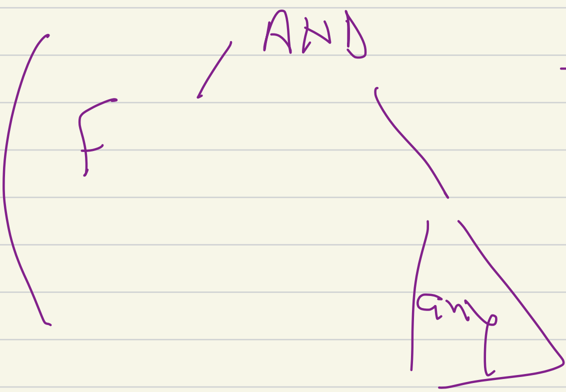



Idea!



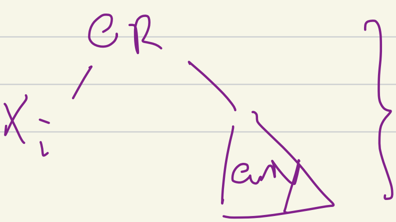

tree, at least one leaf

$x_i \longrightarrow F$




$= F$

this entire tree goes away




$x_i = F$ the tree does not go away

$x_i = T$ the tree goes away

So ... pick $i \in [n]$ s.t.

$x_i$, $\neg x_i$ appear as leaves

at least $\frac{1}{n} \cdot S_n$ times

in the formula.

If you set $x_i = T$ and $x_i = F$ } each time the $x_i$ leave goes away, the time the tree on the other side goes away

$x_i$ tree!

$\geq 1$ leaf

So average # leaves eliminated $\geq \frac{1.5}{n}$

If so, for some $i \in [n]$, setting $x_i$ to T or to F eliminates $\frac{1.5}{n}$ of the leaves

So

1.5 ~~> 1.55

$$S_{n-1} \le \left(1 - \frac{1.5}{n}\right) S_n$$

$S_n \ge n^{1.5}$

$S_n \ge n^{1.55}$

So

$$S_{n-2} \le \left(1 - \frac{1.5}{n-1}\right) S_{n-1}$$

$$\le \left(1 - \frac{1.5}{n-1}\right)\left(1 - \frac{1.5}{n}\right) S_n$$

$$S_1 \le \left(1 - \frac{1.5}{2}\right)\left(1 - \frac{1.5}{3}\right) \cdots \left(1 - \frac{1.5}{n}\right) S_n$$

$$S_1 = 1 \le \overbrace{\frac{1}{n^{1.5}}\, c \quad S_n}$$

$$S_n \ge 1 \cdot n^{1.5} / c \qquad c = \sqrt{8}$$

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Rem:

$x_1 \oplus \dots \oplus x_n$ set $x_i = \begin{cases} T \\ F \end{cases}$

$\Downarrow$

you get no $x_i$

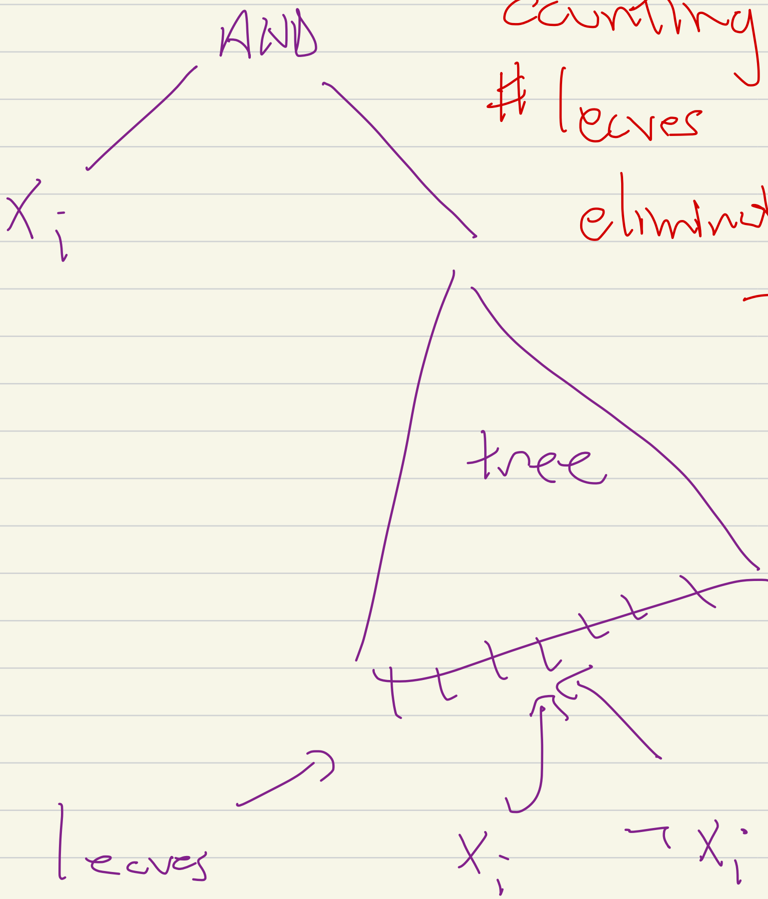
$x_1 \oplus \dots \oplus x_n \qquad \neg(x_1 \oplus \dots \oplus x_n)$

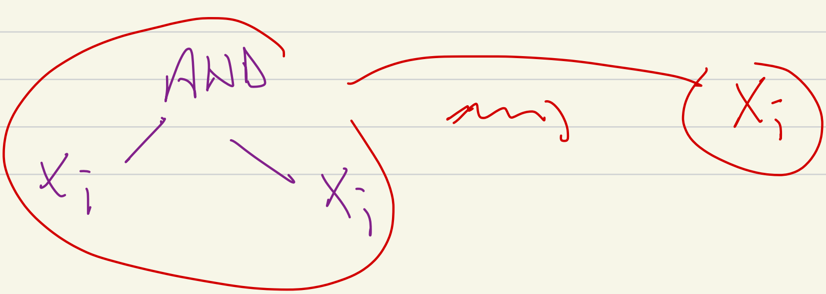
$x_i = F \qquad x_i = T$

But: min formula size for any Boolean function, $g(x_1, \dots, x_n)$, = min formula size $\neg g$

What if

over counting # leaves eliminated?



**Lemma:** [Subbotovskaya]

For any $g : \{T,F\}^n \to \{T,F\}$ :

$$\boxed{x_1 \wedge g(x_1, \dots, x_n) = x_1 \wedge g(T, x_2, \dots, x_n)}$$

$$\neg x_1 \wedge g \dots = \neg x_1 \wedge g(F, x_2, \dots, x_n)$$

$$x_1 \vee g \dots = x_1 \vee g(F, \dots)$$

$$\neg x_1 \vee g \dots = \neg x_1 \vee g(T, \dots)$$

# So what? ...

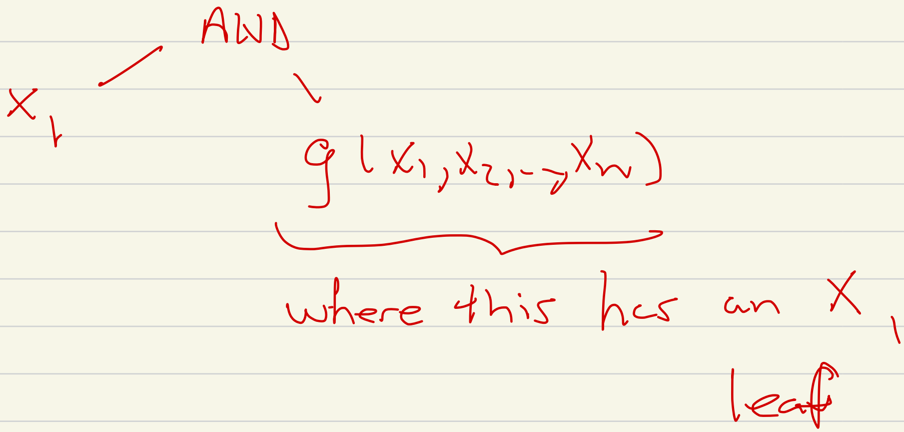
$\Longrightarrow$ for a min size formula

AND

$x_i$

leaf $x_i, \neg x_i$ is impossible

Cor: In a minimum size formula, you can't have




$x_1 = F$,

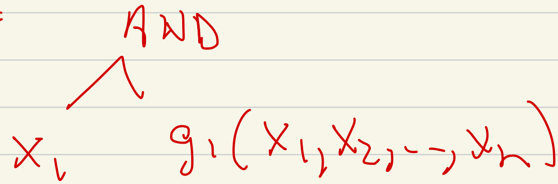
then

$x_1$ and $g_1(x_1, \ldots, x_n) = F$

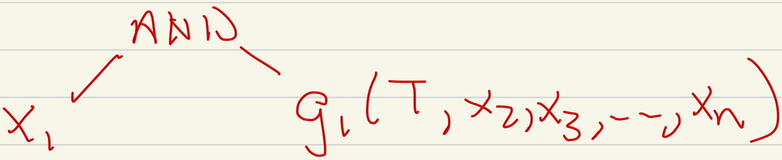
$x_1 = T$

$$\underbrace{x_1 \text{ and } g_1(x_1, \ldots, x_n)}_{g_1(T, x_2, \ldots, x_n)}$$

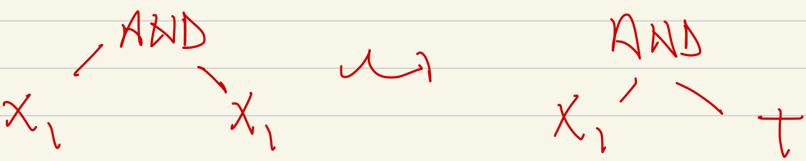
if

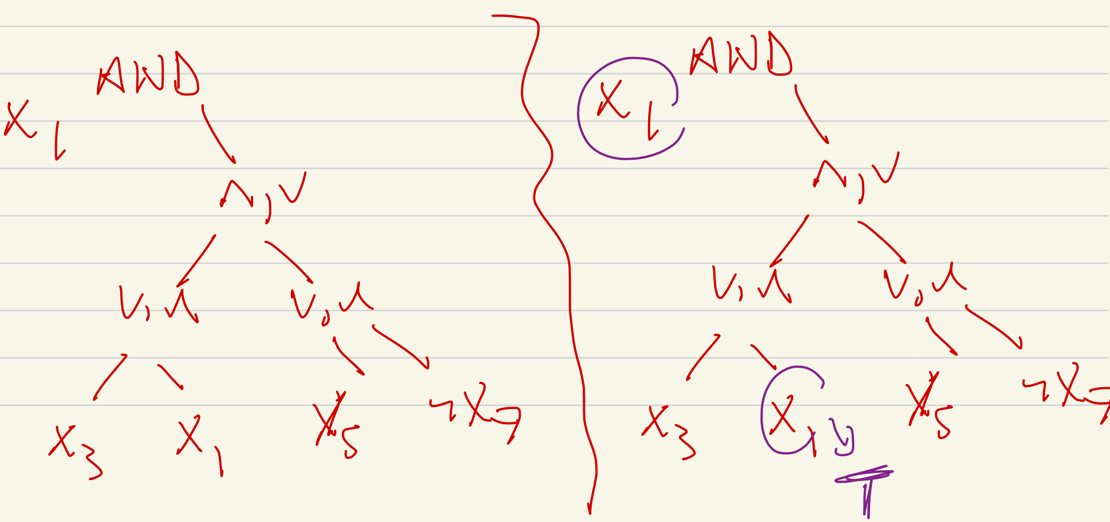



may as well use




e.g.,



Next time (or an exercise)

Andre'ev's function;

Subbotovskaya's method gives

min formula size $\geq n^{2.5}$

$$n^{2.55} \Big\} 1993$$

$\vdots$

$$n^{3-\varepsilon} \Big\} 1998$$