CPSC 421/501 Nov 21, 2025

- Last time: G a graph

$\langle G \rangle \in \Sigma^*_{graph}$,

$\Sigma_{graph} = \{0, 1, \ldots, 9, \#\} = \Sigma_{Tm}$

$\left(\Sigma_{bool} = \{0, \ldots, 9, \#, (,), x, \ldots\} \right)$ = etc. = Σ_{DFA}

- Today: 3COLOUR $\in$ NP

3SAT $\in$ NP

- Start: If 3SAT $\in$ P, then P = NP (Cook-Levin Theorem).

- Hence P = NP $\iff$ 3SAT $\in$ P

Homework 9: $\leadsto$ 10^6 USD — $P \overset{?}{=} NP$

Start: Boolean algebra review,

3CNF form, etc.

AND	F	T
F	F	F
T	F	T

AND $\leftrightarrow \wedge$ $\approx A$

$\leftrightarrow$ Conjunction

OR, $\neg$, XOR,

NAND

Variables $x_1, \ldots, x_n$

Literal $x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n$

3CNF: $C_1 \wedge \ldots \wedge C_m$, $C_i = l_{i_1} \vee l_{i_2} \vee l_{i_3}$

Why 3CNF? Ans: $\left\{ \begin{matrix} \text{3SAT} \\ \text{SAT} \end{matrix} \right\} \rightarrow$ see HW 10

Recall: with $\wedge, \vee, \neg$

and $x_1, \ldots, x_n$ (Boolean vars, T, F)

the total number of Boolean functions

on n variables $(x_1, \ldots, x_n)$

$(x_1 : F, T, \quad x_2 : F, T)$

is $\left\{ \text{functions } \{F, T\}^n \longrightarrow \{F, T\} \right\}$

Say: function f:

x_1	x_2	$f(x_1, x_2)$
F	F	T
F	T	T
T	F	F
T	T	T

$x_1 = F$

$x_2 = F$ $\qquad f(F, F) = T$

$\neg x_1$ and $\neg x_2 \Rightarrow f(x_1, x_2) = T$

$\neg x_1 \wedge \neg x_2 \Rightarrow f(x_1, x_2) = T$

$$\underline{\text{if}}\ \underbrace{(\neg x_1 \wedge \neg x_2)}_{p}\ \underline{\text{then}}\ \underbrace{f(x_1, x_2) = T}_{q}$$

if/then:

$p = T$ then $q = T$

$p = F$ then $q \in F, T$

if/then(p, q):

p	q
T	T
F	T
F	F

...

all Boolean formulas ⟶

Say: function f:

x_1	x_2	$f(x_1, x_2)$
F	F	T
F	T	T
T	F	F
~~T~~	~~T~~	T

$f(x_1, x_2)$ (to be true)

($X_1 = T$ and $X_2 = T$)
OR
($X_1 = F$ and $X_2 = T$)
or
($X_1 = F$ and $X_2 = F$)

$(X_1 \wedge X_2)$
$\vee \; (\neg X_1 \wedge X_2)$
$\vee \; (\neg X_1 \wedge \neg X_2)$

Any Boolean function

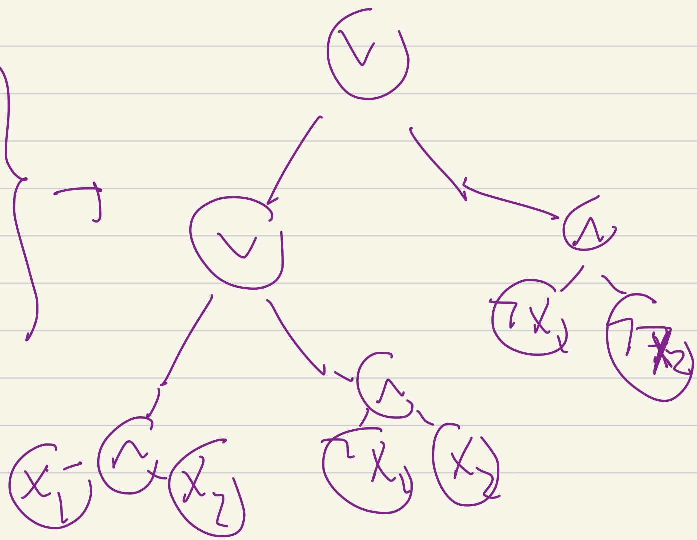
$$f : \{T,F\}^n \to \{T,F\}$$

can be expressed as a formula

in $\wedge, \vee, \neg, (,), x_1, \dots, x_n$

of size at most ...

$$\left.\begin{array}{l} (x_1 \wedge x_2) \\ \vee\ (\neg x_1 \wedge x_2) \\ \vee\ (\neg x_1 \wedge \neg x_2) \end{array}\right\} \neg$$



$\#$ literals

$= \#$ leaves $\overset{\text{here}}{=} 6$

$n = 3$

$$(x_1 \wedge x_2 \wedge x_3) \overset{\text{or}}{\vee} (\quad \neg \text{ --- } \sim$$

at most $n \cdot 2^n$

truth table

x_1	x_2	...	x_n	f
T	T	...	T	T/F
T	T	... T	F	T/F
⋮				

2^n truth values for $x_1, \ldots, x_n$

grab all true values

Total number of functions:

$$\{T, F\}^n \longrightarrow \{T, F\}$$

is ---

$$2^{(2^n)}$$ very large

$\Rightarrow$ most Boolean functions require

$\geq (2^n / n)$ (constant)

Function $f : \{T, F\}^3 \to \{T, F\}$

x_1	x_2	x_3	f
T	T	T	T
T	T	F	T
			T
			⋮
			T

$f(x_1, x_2, x_3) \equiv T$

$(x_1 \wedge x_2 \wedge x_3) \vee (\quad) \vee (\quad) \ldots$

$\underbrace{x_1 \wedge x_2 \wedge x_3}$

3 leaves

3 literals

$x_1 \oplus x_2$ "exclusive" OR

x_1	x_2	$x_1 \oplus x_2$
T	T	F
T	F	T
F	T	T
F	F	F

x_1	x_2	$x_1 \oplus x_2$
1	1	0
1	0	1
0	1	1
0	0	0

$$x_1 \oplus x_2 = (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$$

$$(x_1 \oplus x_2) \oplus (x_3 \oplus x_4) = (x_1 + x_2 + x_3 + x_4) \bmod 2$$

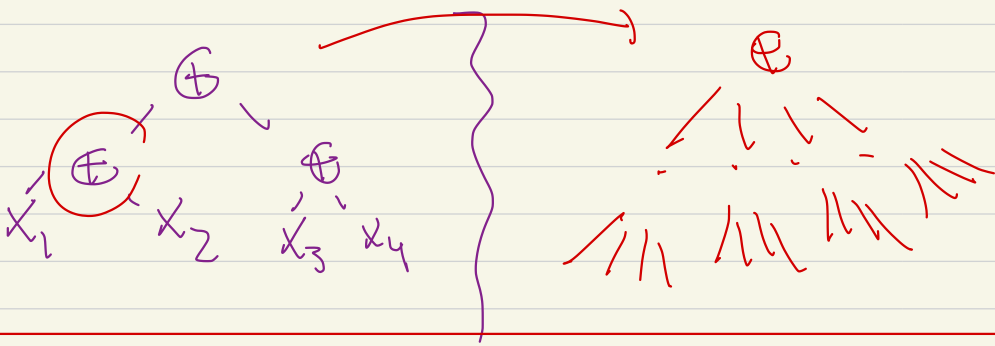
$T = 1$ $F = 0$, $x_1 \oplus x_2 = (x_1 + x_2) \bmod 2$

Claim: There is a formula for

$x_1 \oplus x_2 \oplus x_3 \oplus \dots \oplus x_n$

in terms of $\wedge, \vee, \neg$

of size ... order n^2

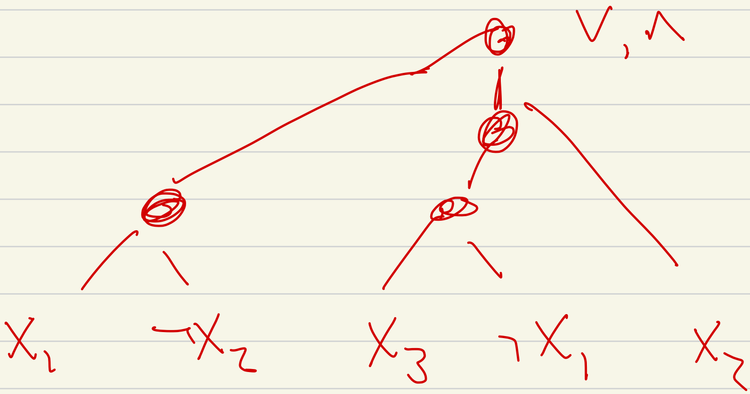


Subbotovskaya (1961) formula for $\oplus$

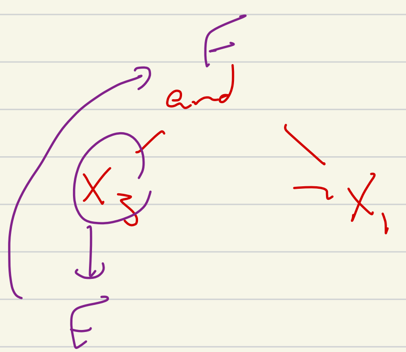
has size at least $n^{1.5}$

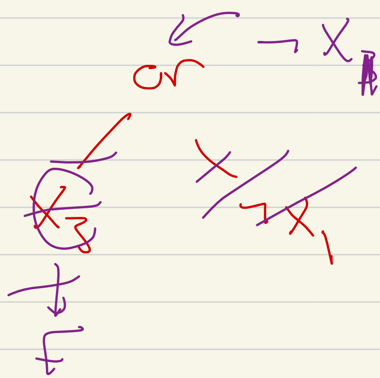
Håstad (1998) ——— $n^2/(\text{very small})$

Class ends



$x_3 \to 0, F$





DFA

initial configuration

↓

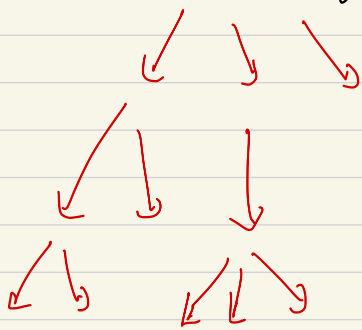
step 2

↓

⋮

NFA

initial configuration



steps 2

steps 3

steps 4

NFA: we accept an input s iff we accept s on at least one possible path

For Turing machines :

3 COLOUR ENP

What is a possible accepting path for a non-deterministic Turing machine on input $s = \sigma_1 \ldots \sigma_n$?

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$$

$$\delta : Q \times \Gamma \to \text{Power}(Q \times \Gamma \times \{L, R\})$$

Answer in a Boolean formula (in 3CNF form)

Answer: (1) Initial config is OK

and

(2)

and

⋮