CPSC 421/501 Sept 10, 2025

→ Bad/good news: any finite document (e.g. your PhD thesis) is merely an element of $ASCII^*$

→ $ASCII^* \xleftrightarrow{\text{bijection}} \mathbb{N} \xleftrightarrow{\text{bijection}} \mathbb{Z} \xleftrightarrow{\text{bijection}} \mathbb{Q}$

Countably infinite

– Countable ⟺ Finite ∪ Countably Infinite

$|ASCII| = 128$, an alphabet

Great news: there is no

surjection:

$$f : \mathrm{ASCII}^* \to \mathrm{Power}(\mathrm{ASCII}^*)$$

Hence take

$$f = \overbrace{\mathrm{LanguageRecognizedBy}}^{\text{we'll explain}}$$

so there exists a language that

is not recognized by any

$$\begin{Bmatrix} \text{Python} \\ \text{C++} \\ \text{APL} \\ \text{Fortran} \\ \text{Haskell} \end{Bmatrix}$$

program, and

we can describe such

a language "explicitly"

Admin

1. You should send a test assignment to gradescope well before 11:59pm on Thursday ...

2. If you aren't yet on gradescope as of today:

   email: jf@cs.ubc.ca

   Subject: __ 421 __ (URGENT)

New today!

a LANGUAGE over Σ (an alphabet) is

- a subset of Σ^*
- an element of $\underbrace{\text{Power}(\Sigma^*)}$

($\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$ strings over Σ)

e.g.

PRIMES = $\{2, 3, 5, 7, 11, \ldots\}$

ODD = $\{1, 3, 5, 7, 9, 11, \ldots\}$

over $\Sigma = \{0, 1, 2, \ldots, 9\}$

PH.D-THESES-UBC-UP-TO-2015

language over ASCII

⋮

finite alphabet

$ASCII^*$

$$ASCII \xrightarrow{\text{bijection}} \{1, 2, \ldots, 128\}$$

$$ASCII^* \xleftrightarrow{\text{bijection}} \{1, 2, \ldots, 128\}^*$$

$$\{1, 2, \ldots, 128\}^*$$

$$= \{ \varepsilon, \ 1, 2, \ldots, 128,$$

$$(1,1), (1,2), \ldots (128, 128),$$

$$(1,1,1), (1,1,2), \ldots \}$$

A set, S, is <u>countable infinite</u>

if there is a bijection

$$f: S \to \mathbb{N} = \{1, 2, 3, \ldots\}$$

OR, equivalently

$$\text{--- --- ---} \quad \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$$

or $\mathbb{Q}$

$$\mathbb{Q} = \{\text{rational numbers}\}$$

$$= \{ \varepsilon, \quad 1, 2, \ldots, 128,$$

$$S_1 \quad S_2 \quad S_3 \quad \cdots \quad S_{129}$$

$$(1,1), (1,2), \ldots (128,128),$$

$$S_{130} \quad S_{131}$$

$$(1,1,1), (1,1,2), \ldots \}.$$

Equivalently, S is countably infinite if

① S is infinite and $\exists$ surjection

$$f : \mathbb{N} \to S$$

② S is infinite and $\exists$ injection

$$f : S \to \mathbb{N}$$

(S, $\mathbb{N} = \{1, 2, 3, \ldots\}$

$$\{1,2\} \xrightarrow{\text{surjection}} \{1\}$$

$$1 \mapsto 1$$

$$2 \mapsto 1$$

Soon we'll see infinite sets

"larger" than $\mathbb{N}$, ASCII*,

such as $\mathbb{R}$, Power($\mathbb{R}$)

uncountably infinite

$\mathbb{C}$

rationals

$\mathbb{R} \subset \mathbb{C}$

$\mathbb{R} \xleftrightarrow{\text{bij}} \mathbb{C}$

Thanks MPW

Soon ...

$$\mathrm{Power}(\{0,1,\ldots,9\}^*) \xrightarrow{\text{injection}} \mathbb{R}$$

Or: 4.2 [Sip]

$\mathbb{R}$ are uncountably infinite

$\mathrm{Power}(\mathrm{ASCII}^*)$ " "

Example: Let's show:

$\mathbb{Q}_{>0}$ is countably infinite

	1	2	3	4	5	
1	$1/1$	$1/2$	$1/3$	$1/4$	$1/5$	---
2	$2/1$	$2/2$	$2/3$	$2/4$	$2/5$	---
3	$3/1$	$3/2$				
4	$4/1$					
5						

etc.

S_1

$1/1,$

S_2 $\qquad$ S_3

$1/2, \quad 2/1$

$1/3, \quad 2/2, \quad 3/1,$

$S_4 \qquad S_5 \qquad S_6$

$S_1, S_2, S_3, S_4, S_5, S_6, \ldots$

S_i = i^{th} rational in seq

$1/1, \ 1/2, \ 2/1, \ 3/1, \ 2/2, \ 1/3, \ \ldots$

$$f : \mathbb{N} \longrightarrow \mathbb{Q}_{>0} = \mathbb{Q}^{+}$$

$$i \mapsto s_i = i^{th} \text{ rational in sequence}$$

$$1/1,\ 2/1,\ 1/2,\ 3/1, \ldots$$

s.t. f is surjective

"Size of $\mathbb{N}$" $\geq$ Size $\mathbb{Q}^{+}$

$$|\mathbb{N}| \geq |\mathbb{Q}^{+}|$$

$1/1,\ 1/2,\ 2/1,\ 1/3,\ \cancel{2/2},\ 3/1,$

$4/1,\ 3/2,\ \ldots$

Say that

1. there is a surjection

$$S \to T$$

2. there is a surjection

$$T \to S$$

Q: Is there a bijection $S \to T$

Homework due to J.F.

inspired by A.H.

Class ends

$$\underbrace{1/1,}_{1} \quad \underbrace{1/2, \; 2/1,}_{2} \quad \underbrace{1/3, \; 2/2, \; 3/1}_{3}$$

Homework S.A. $\rightarrow \binom{2}{2}$

$1+2 = 3 \quad \binom{3}{2}$

$1+2+3 = 6 \quad \binom{4}{2}$

$10 \quad \binom{5}{2}$

15

"triangular numbers"

100th number

$\binom{n}{2} \le 100 \le \binom{n+1}{2}$

$|S| < |T|$

if $\nexists$ surjection $S \to T$

$\nexists$ injection $T \to S$

for us, they're the same

There's no surjection