CPSC 421/501 Homework 8 Solutions 2025

Note: In problems (1)-(3) below we use the notation

$$\mathrm{AccFut}_{L,\Sigma}(s), \quad \mathrm{AccFut}_{L,\Sigma'}(s)$$

to denote the dependence of the set of accepting futures on Σ (Σ', etc.)

Hence

$$\mathrm{AccFut}_{L,\Sigma}(s) \stackrel{\text{def}}{=} \{ s' \in \Sigma^* \mid ss' \in L \}$$

which depends on Σ.

(1)[(a)] Let $Q' = Q \cup \{q_b\}$ where $q_b \notin Q$

Let δ' be given by

(i) $\forall q \in Q$, $\quad \delta'(q, a) = \delta(q, a)$

$$\delta'(q, b) = q_b$$

Hence δ' behaves the same on inputs in $\{a\}^*$, but whenever we encounter a "b" on the input string we move to state q_b.

(ii) $\delta'(q_b, a) = q_b$

$$\delta'(q_b, b) = q_b$$

hence once we enter q_b, we never leave. Hence all we need to do is make q_b a rejecting state (i.e. not an accepting state).

Therefore

$$q_0' = q$$

$$F' = F$$

makes $M' = (Q', \Sigma', \delta', q_0', F')$ accept all strings that M does (i.e. strings in L), and reject any string with a b in it. Hence M' recognizes L as a language over $\Sigma' = \{a,b\}$.

(b) We clearly have, for all $s \in \{a\}^*$,

$$\mathrm{AccFut}_{L,\{a,b\}}(s) = \mathrm{AccFut}_{L,\{a\}}(s)$$

Since

$$\{ s' \in \{a,b\}^* \mid ss' \in L \}$$

must be a subset of $\{a\}^*$ (since if s' has a "b" anywhere in it, then $ss' \notin L$.

Similarly is s has a b in it, then for any $s' \in \{a,b\}$ we have that $ss' \notin L$.

Hence

$$\mathrm{AccFut}_L(s)$$

$$= \{ s' \in \{a,b\}^* \mid ss' \in L \} = \emptyset .$$

Hence the number of sets of the form

$$\{ s' \in \{a,b\}^* \mid ss' \in L \}$$

as s ranges over all $s \in \{a,b\}^*$

is

(1) the number of sets of the form

$$\{ s' \in \{a\}^* \mid s s' \in L \}$$

(with s ranging over all $s \in \{a\}^*$,

(2) plus the set $\emptyset$ (if it is not already counted in part (1)).

- Prob (2) -

(2) In view of the solution to problem (1), each set

$$\mathrm{AccFut}_{L,\{a,b\}}(s) = \{ s' \in \{a,b\}^* \mid s s' \in L \}$$

with $s \in \{a\}^*$ equals

$$\mathrm{AccFut}_{L,\{a\}}(s) = \{ s' \in \{a\}^* \mid s s' \in L \}$$

If L is not regular, then there are infinitely sets of the form

$$\mathrm{AccFut}_{L,\{a\}}(s) = \{ s' \in \{a\}^* \mid ss' \in L \},$$

and hence infinitely many of the form

$$\mathrm{AccFut}_{L,\{a,b\}}(s) = \{ s' \in \{a,b\}^* \mid ss' \in L \}$$

Hence L is non-regular over $\{a,b\}$.

- Prob (3) -

(3) Similarly if $L \subset \Sigma^*$, we have for all $s \in \Sigma^*$

$$\begin{aligned}
\mathrm{AccFut}_{L,\Sigma}(s) &= \{ s' \in \Sigma^* \mid ss' \in L \} \\
&= \{ s' \in (\Sigma')^* \mid ss' \in L \} \\
&= \mathrm{AccFut}_{L,\Sigma'}(s)
\end{aligned}$$

and if $s \in (\Sigma')^* \setminus \Sigma^*$, then

$$\mathrm{AccFut}_{L,\Sigma'}(s) = \emptyset .$$

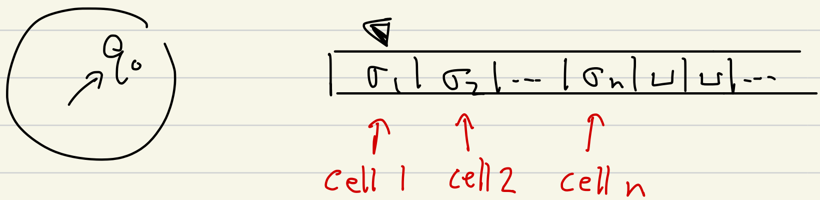
Hence if L is regular, there is between 0 and 1 more accepting futures for L over Σ' than over Σ.

Similarly, if L is non-regular, then the number of accepting futures over Σ for L is infinite, as it therefore is over Σ'.

(4) Note that the general idea is clear; it is more difficult to write a rigorous solution.

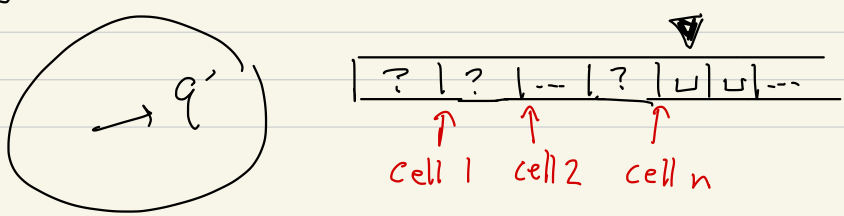
Say that $s \in L$, $s = \sigma_1 \ldots \sigma_n \in \Sigma^n$.

Step 1 (the initial step) looks like



and since we only move R, we either:

(i) reach step n in some state q'



where the contents of cells 1 to n are irrelevant for future states. In this case after enough R moves we must end up in q_{acc}; hence either

$$q' = q_{acc}$$

or

$$\delta(q', \sqcup) = (q_{acc}, \text{irrelevant}, R)$$

or

$$\delta(\delta(q', \sqcup), \sqcup) = (q_{acc}, \text{irrelevant}, R)$$

or etc.

(i.e. after a finite number of R moves over the symbol $\sqcup$ we reach q_{acc}).

The other possibility is that we stop in q_{acc} after step m with $m < n$.

But then let's define

$$\delta(q_{acc}, \gamma) = (q_{acc}, \text{anything}, R)$$

for all $\gamma \in \Gamma$.

Similarly for input $s \notin L$: we either reach q_{rej} or do not halt.

Given such a definition for δ

$$\left(\begin{array}{ll} \text{with} & \delta(q_{acc}, \sigma) = (q_{acc}, \text{anything}, R) \\ & \delta(q_{rej}, \sigma) = (q_{rej}, \text{anything}, R) \\ \text{for all } \sigma \in \Gamma & \end{array} \right)$$

Now we can define the DFA:

$$M = (\tilde{Q}, \Sigma, \tilde{\delta}, \tilde{q}_0, \tilde{F}),$$

namely:

1. $\tilde{Q} = Q$,

2. For all $q \in Q$, $\sigma \in \Sigma$, say that

$$\delta(q,\sigma) = (q', \sigma', R)$$

and set

$$\tilde{\delta}(q,\sigma) = q'$$

(the value of σ' is irrelevant to the Turing machines operation)

3. $\tilde{q}_0 = q_0$

4. $F = \{ q \mid q = q_{acc},$ or

$\delta(q, \sqcup) = q_{acc}$ or

$\delta(\delta(q,\sqcup),\sqcup) = q_{acc}$ or $\ldots \}$

$= \{ q_{acc} \} \cup \{ q \neq q_{acc} \mid$ when starting on q, and we keep moving R over $\sqcup$, we eventually reach $q_{acc} \}$

Hence $s \in \Sigma^*$ is accepted by this DFA iff $s \in L$.

NB: A subtlety here is that after n steps (on an input of size n) we may not necessarily reach q_{acc} or q_{rej}; however, if not, then we must either:

- eventually never reach q_{acc} or q_{rej},

or

- never halt.