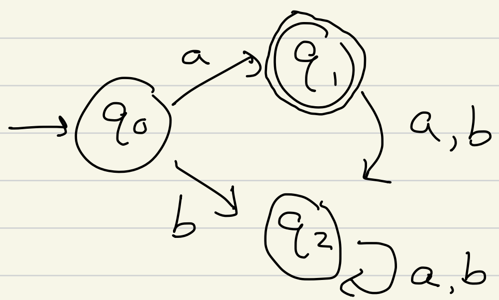
Group Homework 07 Solutions 2025

(c) Joel Friedman

1 (a) (i)

$L_2 = L_1^* = a^*$

First note:



is a 3-state DFA for $L_1 = \{a\}$ over $\Sigma = \{a,b\}$

Hence it suffices to give 3 different values for $\mathrm{AccFut}_{L_1}(s)$ for s varying over $\Sigma = \{a,b\}$; so:

$$\mathrm{AccFut}_{L_1}(\varepsilon) = \{a\}$$

$$\mathrm{AccFut}_{L_1}(a) = \{\varepsilon\}$$

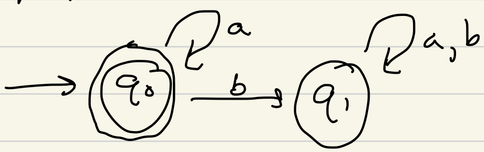
$$\mathrm{AccFut}_{L_1}(a^2) = \emptyset$$

since these sets are visibly different,

Ex 1(a)(i–ii) 2025

the minimum # states in a DFA accepting L_1 is 3.

(a)(ii) L_2 described by a^* is recognized by the DFA



and

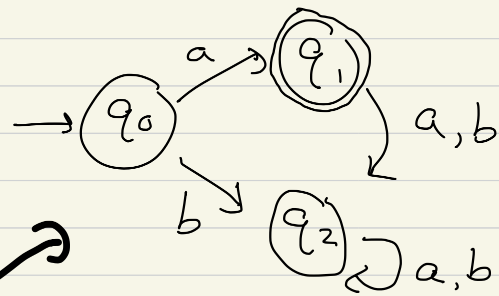
$$\mathrm{AccFut}_{L_2}(\varepsilon) = a^*$$

$$\mathrm{AccFut}_{L_2}(b) = \emptyset$$

Which are visibly different. Hence min # of states in a DFA is 2.

(b)

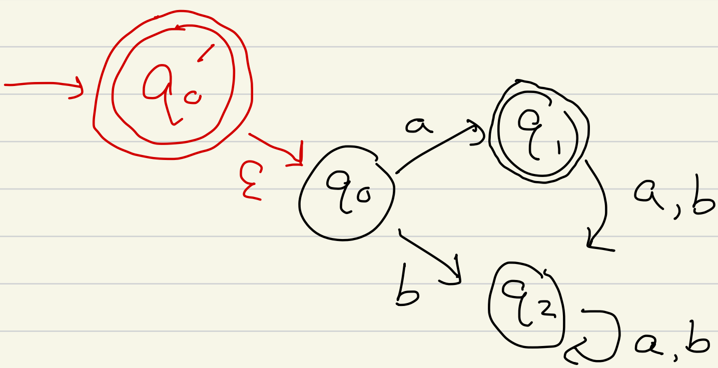
(i) DFA for a:



M

M'

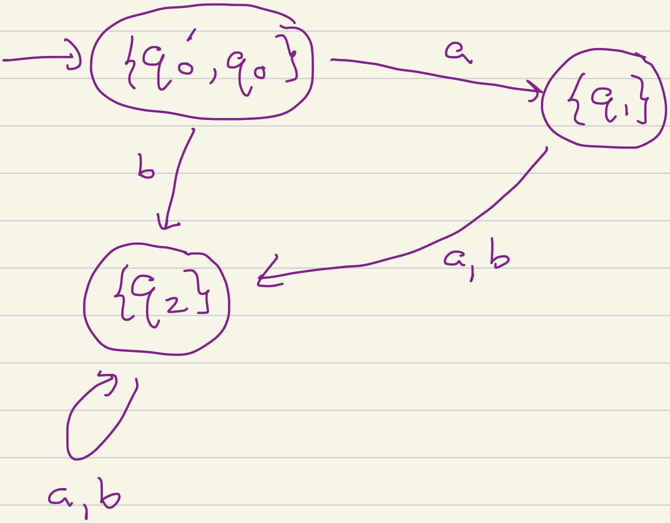
(ii) NFA for a^*:



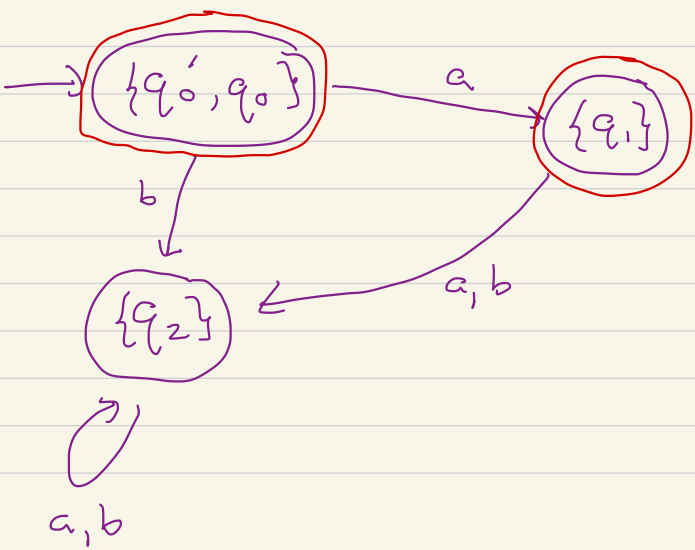
(iii) Since $2^{|Q_{M'}|} = 2^4 = 16$, we'll draw states only as needed. Note: from ε input we can reach $\{q_0', q_0\}$ due to ε jump.

– Ex 1(b)(iii), 2025 –

Hence: STEP 1: States + Transitions



STEP 2: Final states must contain either q_0' or q_1 (final states in the DFA)



→ Ex 2

2. Note: there is a good chance you want parts (a) and (b) in parallel, or even (b) first, if time is of essence...

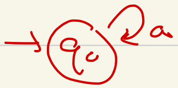
$$a^*b^*b^*a^* = a^*b^*a^* = \{ s \mid \text{all } b\text{'s in } s \text{ occur in one consective block} \}$$

this is for conceptual understanding...

It is essential, but I'd recommend it...

(b) $AccFut_L(\varepsilon) = a^*b^*a^*$

$AccFut_L(a) = a^*b^*a^*$ this is a repeat...

Hence 

$AccFut_L(b) = b^*a^*$ hence a new state!

$AccFut_L(bb) = b^*a^*$ same state as before...

Hence 

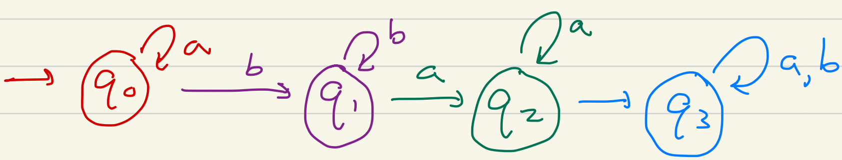
$AccFut_L(ba) = a^*$

$AccFut_L(baa) = a^*$

$AccFut_L(bab) = \emptyset$

$AccFut_L(babs) = \emptyset$ any $s \in \Sigma^*$

Similar pattern

(a) 

(3) In 2025 we built minimum size DFAs for $L = L_k$ with $k = 1, 2$. Hence it should not be a surprise, in the way we stated these examples that:

<u>Claim 1</u> If $\sigma_1, \ldots, \sigma_k \in \{a,b\}$, then $\mathrm{AccFut}_L(\sigma_1 \ldots \sigma_k)$ are all distinct.

AND

<u>Claim 2</u>: If $\tilde{s} \in \{a,b\}^*$, then $\mathrm{AccFut}_L(\tilde{s})$ equals to one of the 2^k accepting futures in Claim 1.

Here we write out (or indicate) formal proofs. You don't need this level of formality to get full marks, but this is judgement call on how closely your exposition demonstrates the knowledge of what is to follow.

<u>Proof of Claim 1:</u> If $\sigma = \sigma_1 \cdots \sigma_k$ and $\sigma' = \sigma_1' \cdots \sigma_k'$ and $\sigma \neq \sigma'$, then for some $i \in [k]$, $\sigma_i \neq \sigma_i'$, i.e. in a diagram

$$\begin{array}{l} \sigma_1 \cdots \sigma_{i-1}\, \sigma_i\, \sigma_{i+1} \cdots \sigma_n \\ \\ \sigma_1' \cdots \sigma_{i-1}'\, \sigma_i'\, \sigma_{i+1}' \cdots \sigma_n' \end{array}$$

$\sigma_i \neq \sigma_i'$

Hence if s' is any string of length $i-1$ ($i-1 \geq 0$, s' exists), then the k^{th} last symbol of $\sigma s'$ and that of $\sigma' s'$ are not equal, and hence one of

$\sigma s', \sigma' s' \in L$, the other is not. Hence

$$\mathrm{AccFut}_L(\sigma) \neq \mathrm{AccFut}_L(\sigma').$$

<u>Proof of Claim 2</u>: We divide $s \in \Sigma^*$ into 3 cases: in each case we find $\tilde{s} \in \Sigma^k$ such that

$$\mathrm{AccFut}_L(s) = \mathrm{AccFut}_L(\tilde{s}) \qquad (*)$$

Case 1: $|s| = k$, then $s \in \Sigma^k$, so we may take $\tilde{s} = s$, and $(*)$ holds.

Case 2: $|s| > k$ and hence $s = s_0 \tilde{s}$ for some $s_0 \in \Sigma^{n-k}$ and $\tilde{s} \in \Sigma^k$. Clearly $(*)$ holds (look at the proof of

Claim 1).

Case 3: $|s|<k$. Since all elements of $L=L_k$ are of length at least k, [by the same idea as in the homework on cases $k=1,2$] we have

$$\mathrm{AccFut}_L(s) = \mathrm{AccFut}_L(b^m s)$$

where $m = k-|s|$.

Since $b^m s \in \Sigma^k$, we may take $\tilde{s} = b^m s$, and hence $(*)$ holds.

Claim 1 shows that there at least 2^k possible values of $\mathrm{AccFut}_L(s)$, and

Claim 2 shows that there are no more than 2^k.

Hence, by the Myhill-Nerode theorem, the minimum number of states in a DFA recognizing $L = L_k$ is 2^k.

[Bonus Question on the next page]

– Ex 4! Bonus Question 2025 –

(4) The Bonus Question (worth 20% above the homework in 2025), regarding giving an algorithm to implement an NFA efficiently, will not be on the exam since I am not releasing a solution.