CPSC 421/501 HW5 2025

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(1)

Given a string $p \sigma_0 i$, let q be the program that accepts j iff p is a valid Python program that on input i does not halt after at most $|j|$ steps (using a universal Python program).

(If p does not accept i after $|j|$ steps then q can reject, or it can loop indefinitely...)

Hence q accepts all its inputs iff p does not accept i (after any finite number of steps), iff $p \sigma_0 i \in$ LOOPING. Hence if L is recognizable, then so is LOOPING, which is impossible.

(2) (a) True:

$L_1 \times L_2$ is the set of all ordered pairs from L_1 and L_2, of which there $|L_1|$ (for the first element of the pair) times $|L_2|$ (for the second). Hence

$$|L_1 \times L_2| = |L_1| \times |L_2|$$

(b) False: there are many possible counterexamples; e.g. if $L_1 = L_2 = \{a^0, a^1\}$, then $L_1 \circ L_2 = \{a^0, a^1, a^2\}$

(c) True:

$$(L_1 \circ L_2) \circ L_3 = \{ s_1 \circ s_2 \mid s_1 \in L_1, s_2 \in L_2 \} \circ L_3$$

$$= \{ s_1 \circ s_2 \circ s_3 \mid s_1 \in L_1, s_2 \in L_2, s_3 \in L_3 \}$$

which similarly equals $L_1 \circ (L_2 \circ L_3)$

(d) True. Using part (a):

$$|L_1 \times L_2| = |L_1| \cdot |L_2| = |L_2| \cdot |L_1| = |L_2 \times L_1|.$$

(e) False: e.g. $L_1 = \{ a^0, a^1 \}$,

$L_2 = bL_1 = \{ b, ba \}$: $L_1 \circ L_2 = \{ b, ab, ba, aba \}$,

$L_2 \circ L_1 = \{ b, ba, ba^2 \}$.

(3) Say that $n = 7p + a$ and $m = 7q + b$

for some $a, b \in \{0, 1, \ldots, 6\}$ and integers p, q.

Then

$$10m + n = 10(7q + b) + 7p + a$$

$$= 7(10q + b + p) + 3b + a$$

Hence

$$(10m + n) - (3b + a) \text{ is divisible by } 7,$$

so

$$(10m + n) \bmod 7 = (3b + a) \bmod 7.$$

(b) Let $Q = \{q_0, q_0', q_1', \ldots, q_7'\}$, where

q_0 is the initial state (which we reject, since $\varepsilon \notin L$)

and for $i = 0, 1, \ldots, 6$, q_i' means that if w is

the input seen until this point, then $w \bmod 7 = i$

(when $w \neq \varepsilon$) (and w may have leading zeros.

Then

$$\delta(q_0, i) = q'_{i \bmod 7} \qquad (i = 0, 1, \ldots, 9)$$

takes us to the correct state upon reading the first input symbol, and by part (a), for $a = 0, 1, \ldots, 6$ and $n = 0, 1, \ldots, 9$, the formula

$$\delta(q'_a, n) = q'_{(3a+n) \bmod 7}$$

takes an input of the form $w\sigma$ where $w \bmod 7 = a$ and $\sigma \in \{0, 1, \ldots, 9\}$ and transitions to the correct value of $w\sigma \bmod 7$. Hence

$Q = \{q_0, q'_0, q'_1, \ldots, q'_6\}$, Σ, δ as above

initial state q_0, and $F = \{q'_0\}$

is such a DFA (i.e. the input is divisible by 7 iff the input is ~~ta~~taken to q'_0).

(c) (i) it suffices to make q_0 accepting, i.e. take $F = \{q_0, q_0'\}$, since the DFA is taken to state q_0 by an input iff the input equals ε.

(ii) One can "merge" q_0 and q_0', that is let $Q = \{q_0', q_1', \ldots, q_6'\}$ with q_0' the initial state, q_0' the only accepting state, and to restrict δ as above to states $q_0', \ldots, q_6'$, i.e.

$$\delta(q_a', n) = q'_{(3a+n) \bmod 7}$$

(4) No solution provided.

This question will NOT appear on this year's exams (2025).