Homework 3

(0) Joel Friedman

(1) r is of the form

$$\frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots$$ where $d_1, d_2, \ldots$

are arbitrary elements of $\{1,2\}$.

There is no proof that r is rational.

[In fact, this shows that some element of

the form $\frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots$ is irrational;

moreover, the set of sequences $(d_1, d_2, \ldots)$

is uncountable, since this is in bijection with

maps $\mathbb{N} \to \{1,2\}$, which can easily put into

bijection with $\mathrm{Power}(\mathbb{N})$.]

(2) 9.2.6 (a)-(c)

(a) Yes — let's prove the contrapositive:

if f not injective, then for some

$a_1, a_2 \in A$, $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

Hence $g(f(a_1)) = g(f(a_2))$. Hence

$$(gf)(a_1) = g(f(a_1)) = g(f(a_2))$$

$$= (gf)(a_2)$$

So gf is not injective.

Hence, if gf is injective,

then f is injective.

(b) No: If $A = \{x\}$, $B = \{x, y\}$, and $C = \{1\}$, and f and g are given by

$$\begin{array}{ccc} x & \longmapsto & x \\ & & y \\ A & \xrightarrow{f} & B \end{array} \qquad \text{AND} \qquad \begin{array}{ccc} x & \longmapsto & 1 \\ y & \longmapsto & \\ B & & C \end{array}$$

then gf takes x to 1, and hence gf is injective. But $g(x) = g(y) = 1$, so g is not injective.

(c) Yes! by (a), f is injective.

Hence for each $b \in \text{Imag}(f)$ there is a unique $a \in A$ s.t. $f(a) = b$.

Hence $|\text{Image}(f)| = |A| \geq |B|$

$\text{Image}(f) \subset B$ and $|\text{Image}(f)| \geq |B|$, we have $\text{Image}(f) = B$.

If g were not injective, then for some $b_1 \neq b_2$ with $b_1, b_2 \in B$ we would have $g(b_1) = g(b_2)$.

Since $\text{Image}(f) = B$, we have

$f(a_1) = b_1$ and $f(a_2) = b_2$

for some $a_1, a_2 \in A$, and

$a_1 \neq a_2$ since $f(a_1) \neq f(a_2)$.

But then

$$(gf)(a_1) = g(b_1) = g(b_2) = (gf)(a_2)$$

but $a_1 \neq a_2$, so gf is not surjective.

(3) No. Indeed, let

$\mathbb{N} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{N}$ be given by $f(x) = x$

and

$$g(y) = \begin{cases} y & \text{if } y \in \mathbb{N} \quad (\text{i.e. } y \geq 1), \\ 1 & \text{if } y \notin \mathbb{N} \quad (\text{i.e. } y \leq 0). \end{cases}$$

Then $\mathbb{N} \xrightarrow{gf} \mathbb{N}$ takes x to x,

so gf is a bijection, but g is

not injective [since $g(1) = g(0) = 1$].

However, $\mathbb{N}$ and $\mathbb{Z}$ are countably

infinite, and hence can be put in

bijection. So

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with f, g as above (and $A = C = \mathbb{N}$, $B = \mathbb{Z}$)

give a counterexample.

[Note: There are many possible variants on the above. We do know that g must be injective on the image of f, and f must be injective; hence we know that A and $f(A)$ are in bijection via f. So it suffices to choose any injection $f: A \to B$ such that $f(A) \neq B$ and A and B are in bijection ...]