Homework 2, CPSC 421/501, 2025

(0) Joel Friedman

(1) 9.2.8

(a) $f(A) = \emptyset$, and

$f(C) = \{A, B, C\} = P$

(b) Since $A \notin f(A)$, $A \in T$

Since $C \in f(C)$, $C \notin T$

We know $T \neq f(A)$, since $A \notin f(A)$

but $A \in T$. We know $T \neq f(C)$ since

$C \in f(C)$ but $C \notin T$.

(c) Now we know $f(B) = \{A, C\}$, and

have $B \notin f(B)$ so $B \in T$.

Since $C \notin T$ and $A, B \in T$, we have

$T = \{A, B\}$.

(2) 9.4.1 (a)

if $(i,j) \in \mathbb{N}^2$ and $i+j \leq 2$,

then $(i,j) = (1,1)$;

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if $(i,j) \in \mathbb{N}^2$ and $i+j = 3$,

then $(i,j) = (1,2)$ or $(2,1)$

Similarly,
if $(i,j) \in \mathbb{N}^2$ and $i+j = k$ for

some $k = 2,3,4,\ldots$ then

(i,j) is one of finitely many

values, namely

$(1, k-1)$ OR $(2, k-2)$ OR $\ldots$ OR $(k-1, 1)$

Hence, for each $k = 2, 3, 4, \dots$

there are only finitely many

$(i, j) \in \mathbb{N}^2$ such that $i + j = k$.

(And, of course each $(i, j) \in \mathbb{N}^2$

does satisfy $i + j = k$ for some $k = 2, 3, \dots$)

Hence we can write a list of all elements

of $\mathbb{N}^2$, namely

$$\underbrace{(1,1)}_{i+j=2}, \quad \underbrace{(1,2), (2,1)}_{i+j=3}, \quad \underbrace{(1,3), (2,2), (3,1)}_{i+j=4},$$

...

Hence $\mathbb{N}^2$ is in bijection with $\mathbb{N}$.

[Note: There are many equivalent ways of writing this; for example, one can write $\mathbb{N} \times \mathbb{N}$ as a table

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ (3,1) & (3,2) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Then the solution above is just like the proof that $\mathbb{Q}^+$ is countable.]

(b) [Again, there are a number of ways of solving this problem.]

For each $\ell = 3, 4, 5, \ldots$ there are only finitely many triples (i, j, k) such that $i + j + k = \ell$. Hence we can list all triples in order of $i + j + k$, e.g.

$i+j+k=3$: $(1,1,1)$

$i+j+k=4$: $(1,1,2), (1,2,1), (2,1,1)$

$i+j+k=5,6,\ldots$: etc.

(Question for your amusement: how many triples do you get s.t. $i+j+k=\ell$, as a function of ℓ?)

(c) There are many ways to do this...

Here's one:

to each $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$, say

the "rank" of $(n_1, \ldots, n_k)$ is

$n_1 + n_2 + \ldots + n_k$. Then

(1) each element of $\mathbb{N}^k$ has finite "rank", and

(2) for each $r = 0, 1, 2, \ldots$ there are only finitely many elements of $\mathbb{N}^k$ of "rank" equal to r: indeed,

rank $(n_1, \ldots, n_k) = r$ implies

(a) $k \leq r$ (since each element of $\mathbb{N} = \{1, 2, \ldots\}$ has value ≥ 1), and

(b) each n_i has $1 \leq n_i \leq r$.

In view of (a), (b), we have

rank $(n_1, \ldots, n_k) \leq r$ implies

$(n_1, \ldots, n_k)$ lies in

$S_r = [r]^0 \cup [r]^1 \cup \ldots \cup [r]^r$, where

$[r] = \{1, 2, \ldots, r\}$, so S_r is finite.

Hence we can list the elements of $\mathbb{N}^*$ by their rank, listing them as the elements of S_0

" " " S_1

" " " S_2

$\vdots$

giving a list $l_1, l_2, l_3, l_4, \cdots$

s.t. each element of $\mathbb{N}^*$ is found somewhere on the list. Hence this gives a surjection

$$\mathbb{N} \to \mathbb{N}^* \quad (\text{namely } i \mapsto l_i).$$

Hence $\mathbb{N}^*$ is countable.

(3) The bonus question: [this solution is not as complete as for a non-bonus question].

(a) There is a bijection between

$[2]^{\mathbb{N}}$ and $\mathrm{Power}(\mathbb{N})$ which takes

$(a_1, a_2, a_3, a_4, \ldots)$ (hence $a_i \in \{1,2\}$ for all i)

to

$$\{n \in \mathbb{N} \mid a_n = 2\}$$

(or $a_n = 1$ if you prefer).

Hence $[2]^{\mathbb{N}}$ is uncountable, since

$\mathrm{Power}(\mathbb{N})$ is (by Cantor's theorem)

(b) Idea: group the sequence

$$(a_1, a_2, a_3, a_4, \ldots) \in [2]^{\mathbb{N}}$$

into groups of 2 bits:

$$a_1 a_2, a_3 a_4, a_5 a_6, \ldots$$

and map

$$
\begin{aligned}
11 &\mapsto 1 \\
12 &\mapsto 2 \\
21 &\mapsto 3 \\
22 &\mapsto 4
\end{aligned}
$$

(or something similar...)

to get a bijection; e.g.

$$1\,1\,1\,2\,2\,2\,2\,1\,1\,1\,1\,2 \ldots$$

$$\downarrow$$

$$11, 12, 22, 21, 11, 12, \ldots$$

$$\downarrow$$

$$1 \quad 2 \quad 4 \quad 3 \quad 1 \quad 2 \quad \text{etc.}$$

(c) Similarly group the bits (binary digits) into groups of size 3.

(d) Reverse the bijection $[2]^{\mathbb{N}} \to [4]^{\mathbb{N}}$ and apply the bijection $[2]^{\mathbb{N}} \longrightarrow [8]^{\mathbb{N}}$ (hence each group of 3 elements of $[4]$ is mapped to a group of 2 elements of $[8]$).

(e) Take $f: [8] \to [7]$ with $f(i) = i$ for $i \leq 7$, and $f(8)$ arbitrary

(Question: how many surjections $[8] \to [7]$ are there?)

The map taking

$$(a_1, a_2, a_3, \ldots) \mapsto (f(a_1), f(a_2), \ldots)$$

gives such a bijection

(f) Similarly.

(g) Compose the surjections

$$[7]^{\mathbb{N}} \to [2]^{\mathbb{N}} \quad \text{and} \quad [2]^{\mathbb{N}} \longrightarrow [8]^{\mathbb{N}}$$

(the second function is also a bijection, but any bijection is also a surjection).

Not for credit: ?? You can set up a map $[7]^{\mathbb{N}} \to [0,1]$ taking $(a_1, a_2, \ldots)$ to $.a_1 a_2 a_3 \ldots$, viewed as a base 7 expansion, but you get collisions, e.g. $.16666\ldots = .20$ in base 7. So you could take out these countable many collisions, and do the same for $[8]^{\mathbb{N}}$

(Q: What about a bijection $[7]^{\mathbb{N}} \to \{0,1\}$??

These sets have the same cardinality, in any system of axioms where S and $S \times \{1,2\}$ have the cardinality for S infinite, such as ZFC.)

(4) (a) The set of $i, j \in \mathbb{N}$ such that $i+j=l$ for l fixed is

$$\{(1, l-1), (2, l-2), \ldots, (l-1, 1)\}$$

which has $l-1$ elements. The number of pairs (i, j) with $i+j \le l$ is

$$1+2+\ldots+l-1 = \binom{l}{2}.$$

So if $i+j=a$, there are $\binom{a-1}{2}$ fractions i'/j' with $i'+j' \le a-1$ that appear on the list strictly befor i/j, and i/j occurrs at latest in position $\binom{a}{2}$.

(b) $\binom{a-1}{2}$ fractions i'/j', occur with $i'+j' \leq a-1$. Since we are ordering the factions on level a as

$$1/a-1, \quad 2/a-2, \ldots, \quad a-1/1$$

the fraction i/j is the i^{th} fraction on this list, and hence occurs as S_k where

$$k = \binom{a-1}{2} + i = \binom{i+j-1}{2} + i$$

$$= \frac{(i+j-1)(i+j-2)}{2} + i$$

(c) The relative density of reduced

fractions in the way we have listed them is well-knew to be

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots$$

intuitively, (i,j) is reduced iff

(1) i,j are not both divisible by 2, which occurs with probability $\frac{3}{4} = 1-\frac{1}{2^2}$, since both are divisible by 2 with probability $\frac{1}{2^2}$

AND

(2) - - - - 3

- - - - - - $1-\frac{1}{3^2}$

AND

etc.,

and these probabilities are "sufficiently close to being independent" so that the probability of all these events holding is just the product, Of course, this is <u>not</u> a proof. For hints as to how to get a rigorous proof, see Homework 1, page 7, Fall 2019 version of CPSC 421/501 (you can get to the Fall 2019 homepage from this year's homepage). This product is easily seen to be

$$1/\zeta(2), \quad \text{where} \quad \zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}$$

$$= \sum_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)$$

and it is well known that

$$\zeta(2) = \frac{\pi^2}{6}.$$

Hence the relative density is $\dfrac{6}{\pi^2}$, which is well-known to be irrational (even transcendental).

If we had

$$k = \alpha_1 + i\alpha_2 + j\alpha_3 + i^2\alpha_4 + ij\alpha_5 + j^2\alpha_6$$

with $\alpha_1, \ldots, \alpha_6 \in \mathbb{Q}$, then the fraction $1/n$ would occur in position

$$\alpha_1 + \alpha_2 + n\alpha_3 + \alpha_4 + n\alpha_5 + n^2\alpha_6 ;$$

however its true occurrence is in position

$$\binom{n+1}{2}\left(\frac{6}{\pi^2} + o_n(1)\right) = n^2\left(\frac{3}{\pi^2} + o_n(1)\right)$$

which cannot equal $n^2\alpha_6 + O(n)$.

On can take many variants of the faction $1/n$ above.