421/501 Homework 1 Solutions, 2025

(c) Joel Friedman

---

(1) 9.2.2. (a) Since $T = \{ s \mid s \notin f(s) \}$

and $1 \in f(1)$, $1 \notin T$. Since $1 \in \{1,2\}$,

$T \neq \{1,2\}$.

(b) No! (there are many possible examples to show that both $2 \in T$ and $2 \notin T$ are possible)

if $f(2) = \emptyset$, then $2 \notin f(2)$ so

$2 \in T$;

if $f(2) = S$, then $2 \in f(2)$ so

$2 \notin T$;

hence both $2 \in T$ and $2 \notin T$ are possible

---

(2) 9.2.3

(a) $a \in S$ but $a \notin f(a)$; hence

$S \neq f(a)$

(b) Similarly $b \notin f(b)$ but $b \in S$

so $S \neq f(b)$, and $c \notin f(c)$ but $c \in S$

so $S \neq f(c)$.

(c) Since $s \notin f(s)$ for $s = a, b, c$,

$T = \{s \in S \mid s \notin f(s)\} = \{a, b, c\}$.

(3) 9.2.5

[There are many possible examples here...]

Let $f(a) = f(b) = f(c) = S = \{a,b,c\}$.

Then

$\{s \mid s \in f(s)\} = \{a,b,c\} = S$

which is in the image of f

(since $f(a) = S$).

---

(4) 9.2.13

(a) Each of Johnny, Moira, Alexis loves themself. David does not love themself. Hence $T = \{\text{David}\}$.

(b) If David does not love themself, then David $\in T$ and $\{\text{People Whom David Loves}\} = \emptyset$.

So David $\notin \{\text{People Whom David Loves}\}$

but David $\in T$

so $T \neq \{\text{People Whom David Loves}\}$.

(5) No. Of course, you get no credit for just saying "No" (unless this appears as a true/false question).

(5, Solution 1) If $f: S \to T$ is a surjection, then $|S| \geq |T|$ (you really to convince yourself that this true when $S = \emptyset$, which is obvious, but you won't lose points if you assumed this...).

Since $|\emptyset| = 0$,

$\text{Power}(\emptyset) = \{\emptyset\}$ (i.e. the set of all subsets of $\emptyset$ includes $\emptyset$ but no other set)

$|\text{Power}(\emptyset)| = 1$

So $$|\emptyset| = 0 < 1 = |\text{Power}(\emptyset)|$$

there is no surjection

$$\emptyset \longrightarrow |\text{Power}(\emptyset)|.$$

(5, Solution 1') This is also OK:

if S is a finite set, and $n = |S|$,

then $|\text{Power}(S)| = 2^n$. Applying this

for $n = 0$ we get

$$|S| = 0 \quad , \quad |\text{Power}(S)| = 2^0 = 1$$

so we get the same conclusion as in

(5, Solution 1). **AGAIN:** this

solution gets full marks, but you

really need to justify $|\text{Power}(\emptyset)| = 1$, and here we are assuming that the evident formula $|\mathrm{Power}(S)| = 2^{|S|}$

when S is finite and non-empty also holds for $S = \emptyset$.

(5, Solution 2) Look carefully at [Sip] (the course textbook by Michael Sipser): a function $f : S \to T$ is a relation on S, T, i.e. a subset of $R \subset S \times T$ such that $\forall s \in S$, $\exists !$ (there exists a unique) r s.t. $(s, r) \in R$. If $S = \emptyset$, and T is any set, then

$S \times T = \emptyset \times T$

$$\overset{\text{by definition}}{=\!=} \emptyset$$

(you might check this by reviewing the definition of Cartesian product $S \times T$)

Hence there is a unique relation, R on $\emptyset, T$, namely $\emptyset$. This relation vacuously satisfies the condition of being a function.

Hence the unique function $\emptyset \longrightarrow T$ is not surjective (unless $T = \emptyset$ ...)

since if $t \in T$, then there is no tuple in R whose second component is t.

Hence no function $\emptyset \to T$ is surjective, assuming $|T| \geq 1$.

Since $\text{Power}(\emptyset) = \{\emptyset\}$ (if you've taken this route, it's pretty clear that you can explain last claim ...), and hence $|\text{Power}(\emptyset)| = 1$,

there is no surjection

$$\emptyset \longrightarrow \text{Power}(\emptyset).$$

Q: Is the unique map $\emptyset \to \emptyset$ surjective?