GROUP HOMEWORK 9, CPSC 421/501, FALL 2025

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**

(0) Who are your group members? Please print if writing by hand.

(1) Give a 1-tape Turing machine that decides $\mathrm{PALINDROME}_{\{a,b\}}$ by modifying and/or completing the diagram in class on October 31. (See class notes that day, especially the last page.) You should (1) describe the high-level ideas, (2) explain the phases and which states are in each phase and what they mean, and (3) explicitly describe δ (either in a diagram, a table, or listing its values).

(2) Give a 2-tape Turing machine that decides $\mathrm{PALINDROME}_{\{a,b\}}$ by modifying and/or completing the diagram in class on November 3. (See class notes that day.) You should (1) describe the high-level ideas, (2) explain the phases and which states are in each phase and what they mean, and (3) explicitly describe δ (either in a diagram, a table, or listing its values).

The next two exercises have some Boolean algebra that we need. We use F, T to denote the Boolean values false and true, and use $\wedge, \vee, \neg$ to denote the logical AND, OR, and NOT respectively; we also respectively refer to these operations as conjunction, disjunction, and negation. Recall that a

Boolean formula in the variables $x_1, \ldots, x_n$ refers to any formula consisting of these variables and the symbols $\wedge, \vee, \neg$ and parenthesis. For example:

$$x_1 \vee (\neg x_2 \wedge x_3) \vee (\neg x_4 \wedge \neg x_1)$$

A Boolean formula is *satisfiable* if there is at least one assignment of the variables to the values T, F which makes the formula true. For example, $x_1 \wedge \neg x_1$ is not satisfiable (it evaluates to F both for $x_1 = F$ and for $x_1 = T$), and $x_1 \wedge \neg x_2$ is satisfiable (it evaluates to T if we set $x_1 = T$ and $x_2 = F$).

(3) Let $n \geq 4$, and let $a_1, \ldots, a_n \in \{F, T\}$. Show that

$$a_1 \vee a_2 \vee \ldots \vee a_n = T$$

iff the formula

$$\phi(z_1, \ldots, z_{n-3}) = \bigl(a_1 \vee a_2 \vee z_1\bigr) \wedge \bigl(\neg z_1 \vee a_3 \vee z_2\bigr) \wedge \ldots \wedge \bigl(\neg z_{n-4} \vee a_{n-2} \vee z_{n-3}\bigr) \wedge \bigl(\neg z_{n-3} \vee a_{n-1} \vee a_n\bigr)$$

is satisfiable.

(4) Recall that a 3CNF (3 conjunctive normal form) Boolean formula in the variables $x_1, \ldots, x_n$ is a Boolean formula of the form $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ where each c_m is of the form $\ell_{m,1} \vee \ell_{m,2} \vee \ell_{m,3}$ where each $\ell_{i,j}$ is a literal, meaning one of $x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n$. [Hence a formula is in 3CNF if it is a conjunction of clauses, each clause being a disjunction of exactly three literals.] For example

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_1 \vee \neg x_2)$$

is in 3CNF form (there can be repeated literals in a clause).

Show that there is a Boolean function $\{F, T\}^4 \to \{F, T\}$ that cannot be expressed as a 3CNF Boolean formula. [Hint: show that a disjunction of three literals is either never false (e.g., $x_1 \vee x_1 \vee \neg x_1$) or false on at least $1/8$ of the $16 = 2^4$ possible assignments of x_1, x_2, x_3, x_4 to $\{F, T\}$.]

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