GROUP HOMEWORK 5, CPSC 421/501, FALL 2025

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Disclaimer: The material may sketchy and/or contain errors, which I will elaborate upon and/or correct in class. For those not in CPSC 421/501: use this material at your own risk...

Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**

(0) Who are your group members? Please print if writing by hand.

(1) Show that

$$L = \{q \mid q \text{ is a valid Python program that accepts all of its inputs}\}$$

is unrecognizable. [Hint 1: We know that LOOPING is unrecognizable. Show that LOOPING can be reduced to L, i.e., LOOPING can be recognized if you have a program that recognizes L.] [Hint 2: Specifically, given $p\sigma_0 i$, let q be the program that on input j simulates p on input i for $|j|$ steps (say, using a universal Python program) and takes an appropriate action based on the result.]

(2) Let L_1, L_2 be finite languages, i.e., as sets, $|L_1|, |L_2| < \infty$. Which of the following are true or false? Explain.[1]
  (a) $|L_1 \times L_2| = |L_1|\,|L_2|$.
  (b) $|L_1 \circ L_2| = |L_1|\,|L_2|$.
  (c) $(L_1 \circ L_2) \circ L_3 = L_1 \circ (L_2 \circ L_3)$.
  (d) $|L_1 \times L_2| = |L_2 \times L_1|$.
  (e) $|L_1 \circ L_2| = |L_2 \circ L_1|$.

[1]This problem inspired by conversions in 2025 with S.A.

(3) Let L be the set of strings of digits, i.e., strings over the alphabet $\Sigma_{\text{digits}} = \{0, 1, \ldots, 9\}$, that represent integers in base 10 that are divisible by 7, where we allow leading 0's but we don't consider the empty string, ϵ, to be part of L. Hence

$$L = \{0, 7, 00, 07, 14, 21, 28, \ldots, 91, 98, 000, 007, 014, \ldots, 098, 105, 112, \ldots\}.$$

Recall that if $n \in \mathbb{Z}$, the expression $n \bmod 7$ refers to the unique integer, $a \in \{0, 1, \ldots, 6\}$ such that $n = 7p + a$ for some $p \in \mathbb{Z}$, and that if n, n' are integers, then $n - n'$ is divisible by 7 iff $n \bmod 7 = n' \bmod 7$.

(a) Show that for any integers $m, n \in \mathbb{Z}$, we have

$$(10m + n) \bmod 7 = \Bigl(3(m \bmod 7) + (n \bmod 7)\Bigr) \bmod 7.$$

(b) Use the previous part to design an 8-state DFA, M, that recognizes L. [Hint: it is easiest to name states in some convenient way so that you can describe the values of $\delta(q, \sigma)$ by a simple formula. You probably don't want to draw a graph that would need to depict $8 \cdot 10 = 80$ transition arrows...]

(c) Say that $L' = L \cup \{\epsilon\}$, so that L' is the language of strings representing integers divisible by 7, where we allow leading 0's and we do allow/consider the empty string ϵ, to be part of L'.

(i) Describe a simple modification of M that yields another 8-state DFA, M',[2] that recognizes L'.

(ii) Can you describe a 7-state DFA that recognizes L' ?

(4) Bonus Question (worth 20% above the homework). Say that we have a family of Python programs $\{p_j\}_{j \in J}$ where J is an arbitrary set (hence J can be countable or uncountable). (In other words, we have a function $f : J \to \text{ASCII}^*$ and for each $j \in J$, $p_j = f(j)$ is a (valid) Python program.) Let

$$L = \{i \in \text{ASCII}^* \mid \text{for some } j \in J,\ i \text{ is accepted by } p_j\}.$$

Is L necessarily recognizable?[3]

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[2] Corrected in 2025 by an anonymous student.

[3] Thanks to A.K. in 2025 for suggesting this type of problem.