GROUP HOMEWORK 2, CPSC 421/501, FALL 2025

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**

(0) Who are your group members? Please print if writing by hand.

(1) Exercise 9.2.8 on the handout "Uncomputability in CPSC421/501."

(2) Exercise 9.4.1, parts (a) and (c) on the handout "Uncomputability in CPSC421/501."

(3) Exercise 9.4.3 on the handout "Uncomputability in CPSC421/501"[1] is no longer part of this problem set. Instead, this homework consists of 3 problems; the following is a bonus problem, worth 20% (for a total possible 120% on this homework).

Recall that for $n \in \mathbb{N}$, $[n]$ denotes $\{1, 2, \ldots, n\}$. For a sets A, B, we use A^B to denote the set of maps from B to A. The justification for this notation is that if A, B are finite (and nonempty, for simplicity), then we have

$$\left|A^B\right| = |A|^{|B|},$$

for example, there are 7^4 maps from $B = [4]$ to the set $A = [7]$.

[1]Thanks to A.H. for discussing this with me on September 10, 2025.

Hence $A^{\mathbb{N}}$ is the set of maps $\mathbb{N} \to A$, which is in one-to-one correspondence with the set of infinite sequences

$$(a_1, a_2, a_3, \ldots) \quad \text{where} \quad a_1, a_2, a_3, \ldots \in A.$$

The point of this exercise is to show that for certain sets S, T, there can be very simple surjections $S \to T$ and $T \to S$, while it seems much harder to describe a bijection $S \to T$.

(a) Is $[2]^{\mathbb{N}}$ countably infinite? Justify your answer.
(b) Describe a simple bijection $f\colon [2]^{\mathbb{N}} \to [4]^{\mathbb{N}}$.
(c) Describe a simple bijection $f\colon [2]^{\mathbb{N}} \to [8]^{\mathbb{N}}$.
(d) Describe a simple bijection $f\colon [4]^{\mathbb{N}} \to [8]^{\mathbb{N}}$, based on your answers to (a) and (b).
(e) Describe a surjection $[8] \to [7]$, and use it to give a surjection $[8]^{\mathbb{N}} \to [7]^{\mathbb{N}}$.
(f) Describe a surjection $[7] \to [2]$, and use it to describe a surjection $[7]^{\mathbb{N}} \to [2]^{\mathbb{N}}$.
(g) Using parts (c) and (f), describe a simple surjection $[7]^{\mathbb{N}} \to [8]^{\mathbb{N}}$.

Not for credit: is it no more difficult to describe a bijection $[7]^{\mathbb{N}} \to [8]^{\mathbb{N}}$???

(4) Consider the rational numbers

(1)
$$s_1 = 1/1,\ s_2 = 1/2,\ s_3 = 2/1,\ s_4 = 1/3,\ s_5 = 2/2,\ s_6 = 3/1,\ s_7 = 1/4,\ s_8 = 2/3, \ldots$$

(see class notes on Sept 10). Note that this list has repeated rational numbers, but has no repeated fractions (before reducing). Fix $i, j \in \mathbb{N}$, and let $k \in \mathbb{N}$ be the unique integer such that s_k is the fraction i/j as it appears in the above list.[2] [For this exercise you can assume that $1 + 2 + \cdots + n = \binom{n+1}{2}$; you don't have to formally prove this by induction.]

(a) Let $a = i + j$, explain why
$$\binom{a-1}{2} < k \le \binom{a}{2}, \quad \text{where} \quad \binom{x}{2} \stackrel{\text{def}}{=} \frac{x(x-1)}{2}$$
(for all $x \in \mathbb{R}$, although here we only need it for $x \in \mathbb{N}$).
(b) Give an explicit formula for k (not s_k) as a function of i, j.
(c) Now say that you reduce the fractions (1) but don't list any repeated occurrence of a rational number. Hence you write the list:

$$r_1 = 1/1,\ r_2 = 1/2,\ r_3 = 2/1,\ r_4 = 1/3,\ (\text{we discard } 2/2)\ r_5 = 3/1,\ r_6 = 1/4,\ r_7 = 2/3,$$
$$r_8 = 3/2,\ r_9 = 1/4,\ r_{10} = 5/1,\ (\text{we discard } 2/4, 3/3, 4/2)\ r_{11} = 5/1,\ r_{12} = 1/6,\ r_{13} = 2/5,\ \ldots$$

Show that there is no polynomial of degree 2, $p(i, j)$, with rational coefficients with the following property: if $i, j, k \in \mathbb{N}$ are such that $r_k = i/j$ (in other words, i/j is a reduced fraction, i.e., i and j have no common divisor greater than 1) then $k = p(i, j)$.

[Hint 1: you **are allowed to ask favourite AI program** what the above has to do with $1/\zeta(2)$, "1 over Zeta of 2"; you can use the result as long you understand the result and are reasonably sure that it's

[2] This problem is new as of 2025, and was inspired by a discussion after class on September 10, 2025 with S.A.

true (if your quoted result is wrong, then you don't get full marks on this part). In this case you must acknowledge your AI program, which version you are using, and what it costs you (not in time, but in money, e.g., the free version, the $20(USD)/month, etc.).]
[Hint 2: You are free to **think about this yourself, without the help of AI; solving the problem this way would use ideas not covered this course**. You might consider the relative density of reduced fractions i/j; if you are running numerical simulations you should acknowledge this.]
[Hint 3: You could first try Hint 1, then check its reasonability using Hint 2.]

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