CPSC 421/501 Dec 1, 2023

- If 3COLOR (or any NP-complete problem) over $\Sigma = \{T, F\}$ has

$$\text{Function}_{3COLOR,n}(x_1, \ldots, x_n)$$

that require circuits of size $\geq Cn^k$ for all C, k, then $\boxed{P \neq NP}$.

- If so, then $\text{Function}_{3COLOR,n}$ requires formulas of size $\geq Cn^k$.

- Let's start there. (... 1950's 1960's $\to$ 2023)

- Let's start with any function $\{T,F\}^n \to \{T,F\}$

Today, Friday Dec 1:

Last day of new material!

Mon, Wed, Dec 4, 6

- 501 Presentations
- Exam practice

Dec 11 → Final Exam

$\text{Parity}_n(x_1, \ldots, x_n)$ is defined as

$(x_1 + \ldots + x_n) \bmod 2$ with $x_i \in \{0,1\}$

equivalently,

x_1 xor x_2 xor ... xor x_n with $x_i \in \{T, F\}$.

<u>Thm (Easy)</u>: Parity_n has formulas of size $\leq C n^2$.

<u>Thm (Subbotovskaya, 1961)</u>: Parity_n requires formula size $\geq c n^{3/2}$ (where $c > 0$).

Fact! A function

$$f : \{T, F\}^n \longrightarrow \{T, F\}$$

OR

$$\{0, 1\}^n \longrightarrow \{0, 1\}$$

$$f = f(x_1, \dots, x_n)$$

(1) If f depends on all $x_1, \dots, x_n$

$$\text{MinFormulaSize}(f) \geq n$$

(2) To prove

$$\text{MinFormulaSize}(f) \geq n^{1\cancel{0}00001}$$

is not easy ... began in 1961 ...

Any $f = f(x_1, \ldots, x_n)$ can be written in formula size

$n \cdot 2^n$ (or $n2^n/2$ or ...)

Most $2^{(2^n)}$ functions $\{T,F\}^n \to \{T,F\}$

require formular size

$$2^n / (4 + \log_2 n)$$

$\text{Parity}_n = \text{Parity}_n(x_1, x_2, \ldots, x_n)$

$= (x_1 + \ldots + x_n) \bmod 2$ $\qquad \{0,1\}$

$= x_1 \text{ xor } x_2 \text{ xor } \ldots \text{ xor } x_n$ $\qquad \{T,f\}$

$$x_1 \oplus x_2 = \begin{cases} T \text{ if } (x_1, x_2) \\ \quad = (T,F) \text{ or } \\ \quad (F,T) \end{cases}$$

$$= (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$$

So

$$f \oplus g =$$

$$(f \wedge \neg g) \vee (\neg f \wedge g)$$

$$x_1 \oplus \dots \oplus x_4 = (x_1 \oplus x_2) \oplus (x_3 \oplus x_4$$

$$(f \wedge \neg g) \vee (\neg f \wedge g)$$

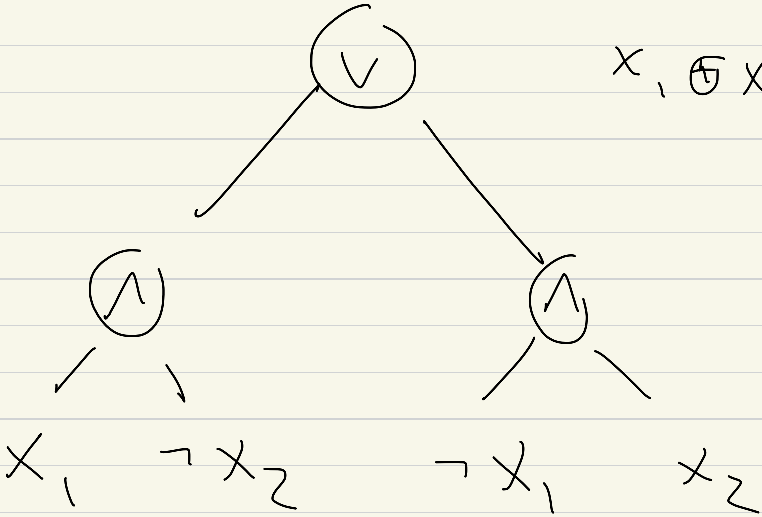
$$f = x_1 \oplus x_2$$

$$g = x_3 \oplus x_4$$

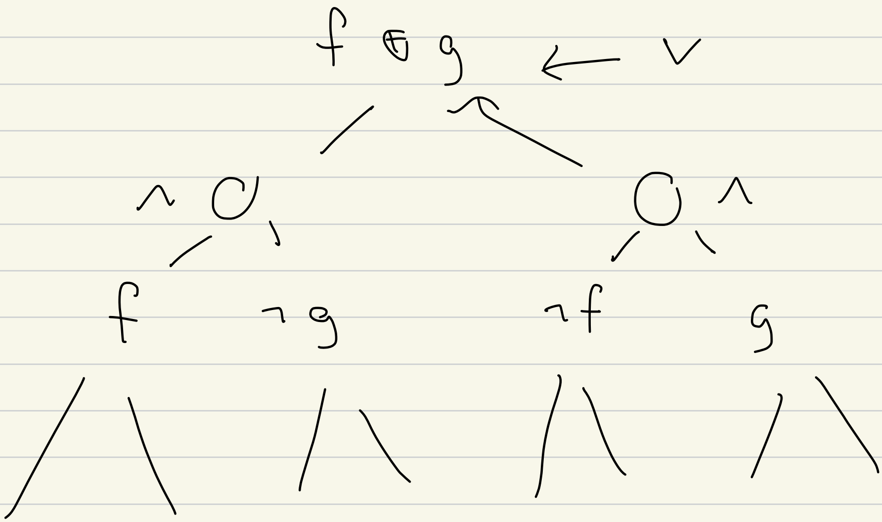
$$x_1 \oplus x_2 \oplus x_3 \oplus x_4$$

$$(x_1 \oplus x_2) \wedge \neg (x_3 \oplus x_4) \vee (\neg x_1 \oplus x_2 \wedge x_3 \oplus x_4)$$

$x_1 \oplus x_2$



same x_3, x_4



Subbotovskaya (1961):

"restriction"

"random restrictions"

Let L_n = min formula size

to express $\text{Parity}_n(x_1, \dots, x_n)$.

Then:

(1) $L_{n-1} \leq L_n \left(1 - \frac{3/2}{n}\right)$

(2) $L_n \geq \left(1 - \frac{3/2}{n}\right)^{-1} L_{n-1}$

$\geq \left(1 - \frac{3/2}{n}\right)^{-1} \left(1 - \frac{3/2}{n-1}\right)^{-1} L_{n-2}$

$$\geq \underbrace{\left(1-\frac{3/2}{n}\right)^{-1}\left(1-\frac{3/2}{n-1}\right)^{-1}\cdots\left(1-\frac{3/2}{2}\right)^{-1}}_{\geq\, n^{3/2}} L_1$$

$$\geq n^{3/2} \cdot 1$$

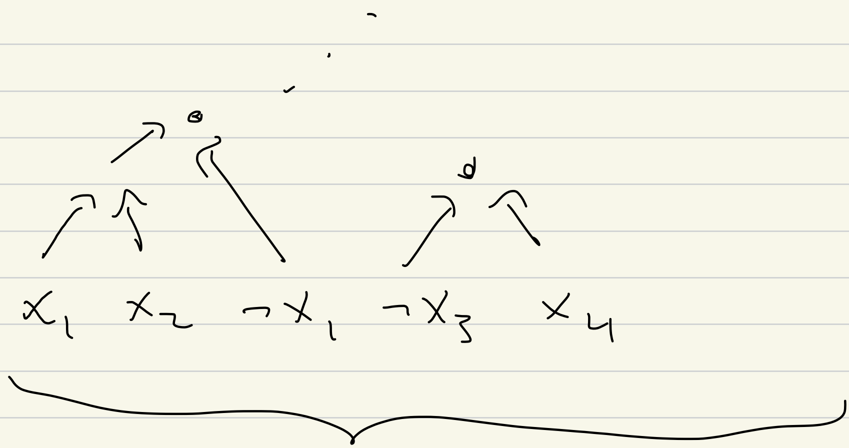
Rem: $\left[\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\cdots\left(1-\frac{1}{2}\right)\right]^{-1}$

$$\left(\frac{n-\cancel{1}}{n}\right)\left(\frac{\cancel{n-2}}{\cancel{n-1}}\right)\cdots\frac{1}{2}\cdots$$

$$= \left[\frac{1}{n}\right]^{-1} = n$$

Pf: If Parity_n has formula size

L_n,



L_n leaves,

then some x_i occurs $\geq \frac{L_n}{n}$ times

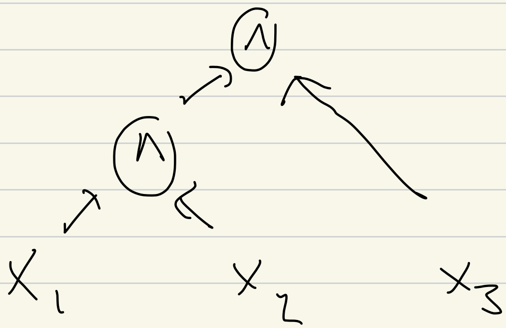
Say x_1:

Set $x_1 = T$, see what happens

" $x_1 = F$, " " "

=

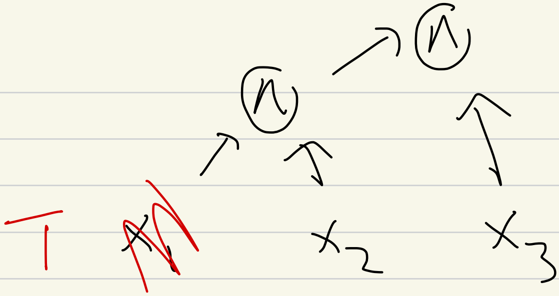
e.g. $x_1 \wedge x_2 \wedge x_3$:

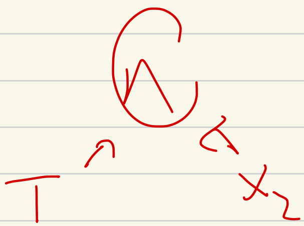


$\leftarrow x_1 \wedge x_2 \wedge x_3$

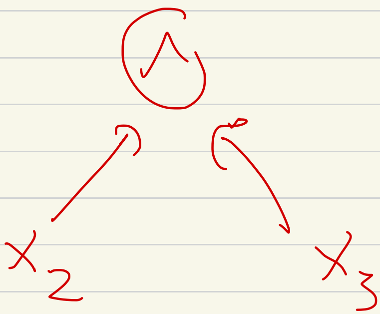
Restrict $X_1 = T$:

we get

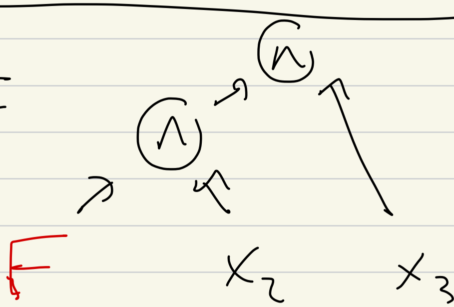




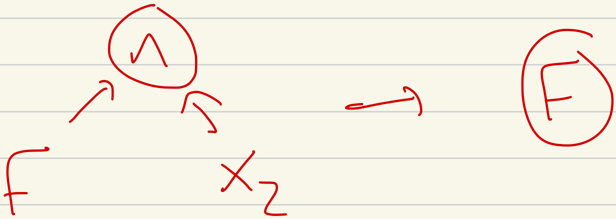
$\rightsquigarrow x_2$

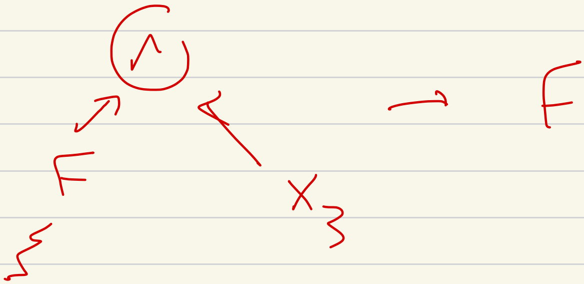


$x_1 = F$



$x_1 = F$:





$=$

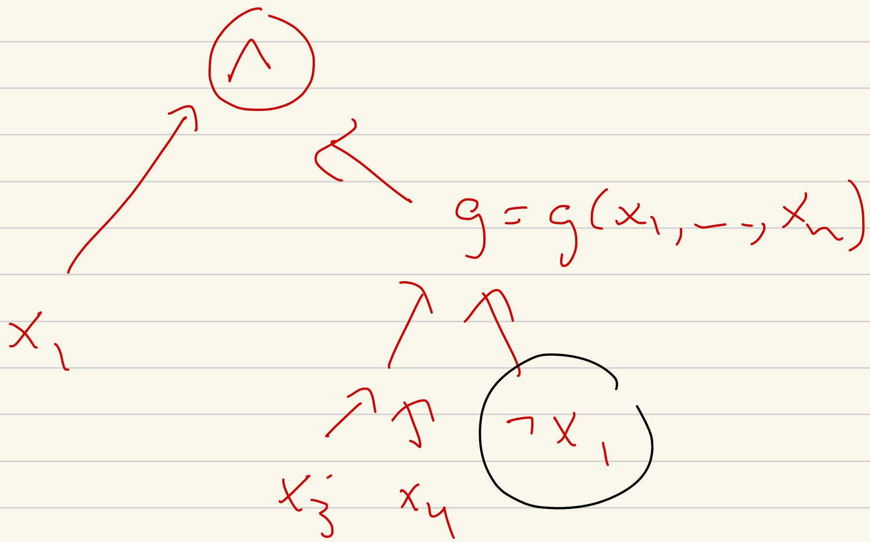
Claim: Subbotovskaya:

by setting $x_1 = T$, $x_1 = F$

on average you eliminate $\geq 3/2$.

(# of the x_1 leaves)

Say



T

if x_1 is on other side, replace T

Lemma:

$$x_1 \wedge g(x_1, \ldots, x_n) = T$$

$$\Longleftrightarrow$$

$$x_1 \wedge g(T, x_2, \ldots, x_n) = T$$

$$x_1 \wedge g = T$$

$$\Longleftrightarrow$$

$x_1 = T$ and $g = T$

$$g = g(x_1, x_2, \ldots)$$

$$\Longleftrightarrow$$

$x_1 = T$ and $g(T, x_2, \ldots, x_n) = T$

$x_1 \vee g = T \iff$

$x_1 = T$ OR $g = T \iff x_1 = T$ or

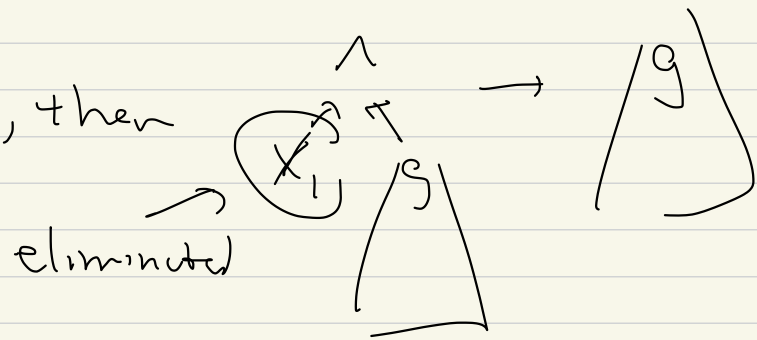
$g = g(F, x_2, \ldots, x_n)$

So if

$$\begin{tikzcd} & \wedge & \\ x_1 \arrow[ur] & & g = g(x_2, \dots, x_n) \arrow[ul] \end{tikzcd}$$

if min size formula

So $x_1 = T$, then



but

$x_1 = F$, then

$$x_1 \wedge g \longrightarrow F \wedge g \longrightarrow F$$

Hence

$x_1 \to T$ eliminates x_1

$x_1 \to F$ " x_1 and least one other leaf

$x_1 \to T$ eliminates 1

$x_1 \to F$ " 2

$\to \geq \frac{3}{2}$ on average !!