CPSC 421/501 Nov 24, 2023

Today: $3SAT \leq_p$ SUBSET-SUM

Given a 3CNF formula,

e.g.

$$f = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3 \vee \overline{x_4})$$

write integers

$n_1, \ldots, n_m, t$

s.t.

$$f \in 3SAT \iff (n_1, \ldots, n_m, t) \in \text{SUBSET-SUM}$$

Next topic: How to solve P vs. NP

Rem! If A, B are NP-complete,

then

$$A \leq_p B$$

and

$$B \leq_p A$$

$$EVEN = \{ w \in \{0, \ldots, 9\}^* \mid w \text{ represents an even integer} \}$$

$$EVEN \in P \subset NP$$

Is EVEN NP-complete?

In 2023, at present, do we know if EVEN is NP-complete?

We want to encode

$x_1 = T$ or F, $x_2 = T$ or F, ...

into integers $n_1, n_2, \ldots$; t

s.t. the subset of integers tells us

$x_1 = T$ or F, $x_2 = T$ or F, ...

and

$$f = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \bar{x}_4) \wedge \ldots$$

equals T on choice of $x_1, \ldots x_n$.

So ...

$c_1 = x_1 \vee \bar{x}_2 \vee x_3$;

we need

$x_1 = T$ OR

$x_2 = F$ OR

$x_3 = T$

$y_1 \leftarrow$ represents $x_1 = T$ $\qquad y_2 \leftrightarrow x_2 = T$

$z_1 \leftarrow$ — — $x_1 = F$ $\qquad z_2 \leftrightarrow x_2 = F$

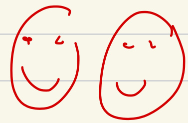
⋮

	$x_1 = T/F$	$x_2 = T/F$			c_1
y_1	1	0	0	---	1
z_1	1	0	0		0
y_2	0	1	0		0
z_2	0	1	0		1
⋮	0	0	1		1
	0	0	1		0
	0	0	0		
	0	0	0		
	0	0	0		
		⋮	⋮		
target	1	1	1		?

1 or 2 or 3

Throw in

$$\begin{array}{lccccc|cccc}
 & 0 & 0 & 0 & \cdots & & 1 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & \cdots & & 1 & 0 & 0 & \cdots \\
 & & & & & & 0 & & & \\
 & & & & & & 0 & & & \\
 & & & & & & \vdots & & & \\
\text{target} & 1 & 1 & 1 & \cdots & 1 & 3 & & &
\end{array}$$



Trick!

We have numbers for subset sum that rep $x_1 = T, F,$ $x_2 = T, F, \ldots$

For clause, we add two extra integers

$$\begin{array}{lccc|cccc}
 & 0 & \cdots & 0 & 1 & 0 & \cdots \\
 & 0 & \cdots & 0 & 1 & 0 & \cdots \\
\text{target} & & & & 3 & &
\end{array}$$

Full idea:

	$x_1 = T/F \quad x_2 = T/F$	C_1	
y_1		1	
z_1		0	
y_2		0	
z_2		1	
$\vdots$		1	
		0	
y_n			
z_n			
aux_1	0 ... 0	1	0 ...
aux_1	0 ... 0	1	0 ...
aux_2	0 ... 0	0	1 0 ...
aux_2	0 ... 0	0	1 0 ...

$\#$ rows $= 2 \cdot n + 2(\#\text{ clauses})$

$\#$ cols $= n + (\#\text{ clauses})$

The size of output is poly in

$n = \#$ varibles and $\#$ clauses

by eliminating needless varaibles from

f,

$$\langle f \rangle \geq n = \# \text{vars}$$

$$\langle f \rangle \geq \# \text{clauses}$$

gives a poly time in $|\langle f \rangle|$

reduction

$$3SAT \leq_p SUBSET\text{-}SUM$$

$$f = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \bar{x}_1)$$

				C_1	C_2
y_1	1	0	0	1	0
z_1	1	0	0	0	1
y_2	0	1	0	0	1
z_2	0	1	0	1	0
y_3	0	0	1	1	1
z_3	0	0	1	0	0
	0	0	0	1	0
	0	0	0	1	0
	0	0	0	0	1
	0	0	0	0	1
target	1	1	1	3	3

f is satisfiable iff

$$\left.\begin{array}{l}1\,0\,0\,1\,0,\\ 1\,0\,0\,0\,1,\end{array}\right\} x_1 = T/F$$

$$\left.\begin{array}{l}0\,1\,0\,0\,1,\\ 0\,1\,0\,1\,0,\end{array}\right\} x_2$$

$$\left.\begin{array}{l}0\,0\,1\,1\,1,\\ 0\,0\,1\,0\,0,\end{array}\right\} x_3$$

0 0 0 1 0,

0 0 0 1 0,

0 0 0 0 1,

0 0 0 0 1,

1 1 1 3 3 $\leftarrow$ target