CPSC 421/501 Sept 8, 2023

"In every math class there is an expert on the empty set."

Heard in 1980, by George Mackey, 1916-2006

Last time:

Alphabet, symbol [letter], string [word]

$\Sigma, \Sigma^k, \Sigma^* = \bigcup_{k=0}^{\infty} \Sigma^k$

Set S: $\text{Power}(S) = \{\text{all subsets of } S\}$

Today:

Define: "Decision Problem", "Algorithm"

Build a problem for which there is no algorithm

<u>Cantor's Thm</u>: If $f: S \to \text{Power}(S)$, then

$$T = \{ s \in S \mid s \notin f(s) \}$$

is not in the image of f.

Generalization: Let $f: S' \to \text{Power}(S)$ be a function (S', S sets) and $g: S \to S'$ a surjection. Then

$$T = \{ s \in S \mid s \notin f(g(s)) \}$$

is not in the image of f.

Subsection 2.5 handout:

~~Un~~ Uncomputability in CPSC 421/501

3 profs $\{A, B, C\}$ say

that

A thinks that $\{A,B,C\}$ are boring

B " " no one is boring

$\emptyset$ are boring

C " " $\{A,B\}$ " "

[C " " C is not boring]

$$f : \{A,B,C\} \to \mathrm{Power}(\{A,B,C\})$$

$$\boxed{\begin{aligned} & f(A) = \{A,B,C\}, \\ & f(B) = \emptyset \quad , \quad f(C) = \{A,B\} \end{aligned}}$$

$f(x) = \{$ subset of $\{A, B, C\}$ that prof x views as boring $\}$

Let

$T = \{ p \in \{A, B, C\}$ s.t.

p thinks that p is not boring,

i.e. $p \notin f(p)$ $\}$

$T = \{ B, C \}$

$p \notin f(p)$

$\boxed{A \in f(A) = \{A, B, C\}}$ no

$B \notin f(B) = \emptyset$ yes

$C \notin f(C) = \{A, B\}$ yes

Why is $T \neq f(A)$

$A \qquad f(A) = \{A, B, C\}$

$T = \{\ \ B, C\}$

$\underline{A \in f(A)} \qquad \underline{A \notin T}$

Centers all students in 1st row

Example:

Oppenheimer	Student 1: no
Barbie	Student 2: no
2001: Space Odyssey	Student 3: yes
Plays Piano	Student 4: no
Plays Guitar	Student 5: no
Born Before 1970	Student 5: no
Taller than 7.ft	Student 6: no
Do you have brown eyes?	Student 6: yes

S ~~Oppenheimer~~ → ~~Student 5~~ S'

Map: $f: S \to S'$

Function: "

$f: \quad S \longrightarrow S'$

$\{$Opp, Bar, Zod, ..., brown eyes$\} \to \{1, 2, ..., 6\}$

Create student (imaginary) that can't be in 1st row!

Student: X

X: has seen Opp, born before 1970

" " Barb

has not " 2001

plays guitar and piano,

more than 7 ft tall,

does not have brown eyes,

We have

(1) Surjection $g: S \to S'$

Image of $g \overset{def}{=}$

$$\{ g(s) \mid s \in S \}$$

the set of things of the form $g(s)$ such that $s \in S$

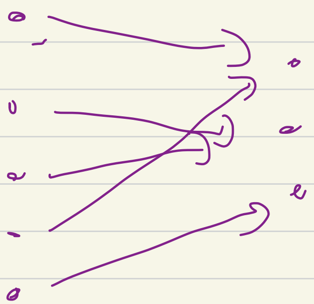
$=$

g is $\left\{ \begin{array}{c} \text{Sujective} \\ \text{A surjection} \\ \text{Onto} \end{array} \right\}$ means $\text{image}(g) = S'$

Intuition: If you have a surjection $g: S \to S'$, then

size of S $\geq$ size of S'

$$|S| \geq S'$$



$$g: S \to S'$$

For any $s \in S$

we ask does $g(s)$ have

property s.

Student 1:

no {Opp, Bub, 2001, 1970, $\overset{\text{taller}}{7 \text{ ft}}$, ne}

yes {piano, guitar}

Student 1 $\longrightarrow$ {$\overset{\text{yes}}{\text{piano}}$, $\overset{\text{yes}}{\text{guitar}}$}

Student 1 $\longmapsto$ {$\overset{\text{yes}}{\text{piano}}$, $\overset{\text{yes}}{\text{guitar}}$}

an elt of S'

$\boxed{f : S' \longrightarrow \text{Power}(S)}$

and $\boxed{g : S \longrightarrow S'}$

Student 1 $\longmapsto$ $\{$ yes piano, yes guitar $\}$

an elt of S'

Make sure student X has

yes to some subset of S

that ensures X = Student 1

$$T = \{ s \mid s \notin f(g(s)) \}$$

s.t. T cannot be anything in image(f)

Example:

$$S' = S$$

[This shows that Cantor's Thm is a special case of Generalized Cantor's Theorem]

$$f : S' \longrightarrow S$$

$$g : S \xrightarrow{\text{identity}} S$$

$$T = \{ s \mid s \notin f(g(s)) \}$$

$$= \{ s \mid s \notin f(s) \}$$