CPSC 421/501 Homework 6 solutions 2023

Individual Homework

(1a) Since 5^n is an odd number for all n (indeed, $5^0=1$, and 5^n for $n \geq 1$ ends in a 5), for any $\sigma_1 \ldots \sigma_m \in \Sigma^*_{\text{five}}$ we have

$$\sigma_1 \ldots \sigma_m \in L \iff$$

$$(\sigma_m 5^0 + \sigma_{m-1} 5^1 + \ldots + \sigma_1 5^{m-1}) \bmod 2 = 0$$

$$\iff (\sigma_m + \sigma_{m-1} + \ldots + \sigma_1) \bmod 2 = 0.$$

Hence it suffices to keep track of the value of $(\sigma_1 + \ldots + \sigma_m) \bmod 2$.

Since we read the input left to right, upon reading σ_k, knowing the value of $(\sigma_1 + \ldots + \sigma_{k-1}) \bmod 2$, we add σ_k to the result, then take the result mod 2, to determine $(\sigma_1 + \dots + \sigma_k) \bmod 2$

[Note: your solution does not need to be so long, but it should include all the main points, such as

(1) $\sigma_1 \dots \sigma_m \in L \iff (\sigma_1 + \dots + \sigma_m) \bmod 2 = 0$

and

(2) we can determine $(\sigma_1 + \dots + \sigma_k) \bmod 2$ knowing $(\sigma_1 + \dots + \sigma_{k-1}) \bmod 2$ and σ_k]

(1b) We have 2 states, q_{even}, q_{odd}, where upon reading $\sigma_1 \dots \sigma_k$ we have

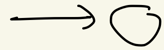
$(\sigma_1 + \dots + \sigma_k) \bmod 2 = 0$ (state q_{even})

OR

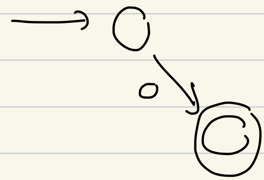
$(\sigma_1 + \dots + \sigma_k) \bmod 2 = 1$ (state q_{odd})

Furthermore, we need to

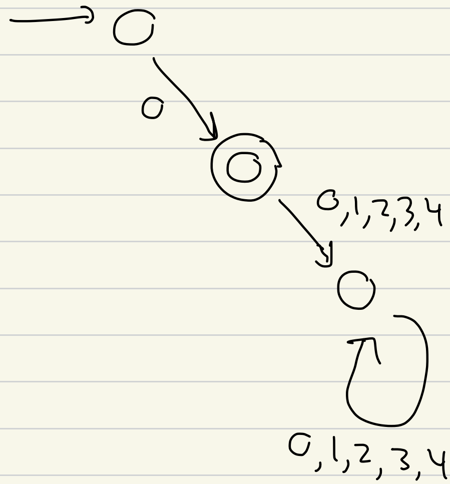
① reject ε, so



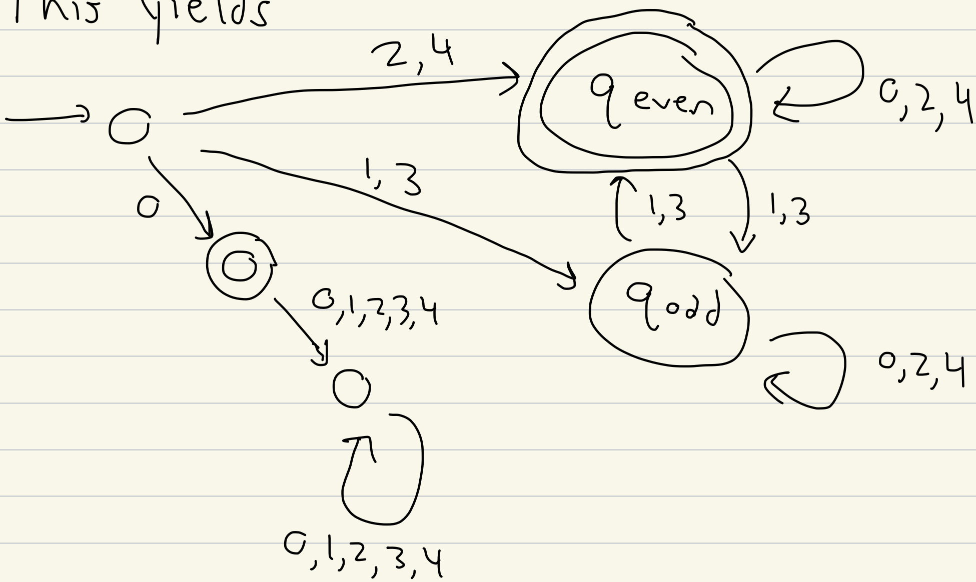
② accept 0, so



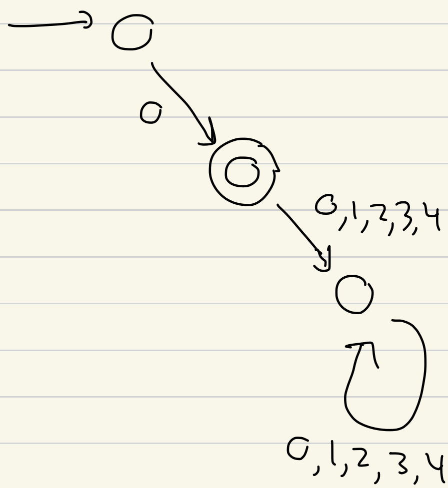
③ reject any other string beginning in 0



This yields



[Again, you don't have to be so wordy, but you should explain the role of q_{even}, q_{odd} and of the extra states



needed for the special cases of ε, 0, and anything with leading 0's.]

Group Homework 6 :

(1) (a) If a DFA, M, recognizes L_1, then by reversing the accepting/rejecting states we recognize the complement, $\Sigma^* \setminus L_1$.

(b) Since the union of two regular languages is regular [in class we explained that the operations $\cup, \circ, *$ preserve regularity],

L_1, L_2 regular $\Rightarrow$ $\Sigma^* \setminus L_1, \Sigma^* \setminus L_2$ regular

$\Rightarrow$ $(\Sigma^* \setminus L_1) \cup (\Sigma^* \setminus L_2)$ is regular

Now we use the fact that the complement of $(\Sigma^* \setminus L_1) \cup (\Sigma^* \setminus L_2)$

is $L_1 \cap L_2$.

[It's OK to merely state this; a formal proof is that

$$(\Sigma^* \setminus L_1) \cup (\Sigma^* \setminus L_2) = \left\{ w \;\middle|\; \begin{array}{l} w \notin L_1 \text{ OR} \\ w \notin L_2 \end{array} \right\}$$

So the complement is

$$\{ w \mid \neg((w \notin L_1) \text{ OR } (w \notin L_2)) \}$$

de Morgan

$$= \{ w \mid (\neg(w \notin L_1)) \text{ AND } (\neg(w \in L_2)) \}$$

$$= \{ w \mid (w \in L_1) \text{ AND } (w \in L_2) \}$$

$$= L_1 \cap L_2 \text{ .}]$$

Hence $(\Sigma^* \setminus L_1) \cup (\Sigma^* \setminus L_2)$ is regular

$\Rightarrow$ $L_1 \cap L_2$ is regular.

(c) L_1, L_2 regular $\Rightarrow$

$L_1, \Sigma^* \setminus L_2$ regular $\Rightarrow$

$L_1 \cap (\Sigma^* \setminus L_2)$ regular (part (b))

and $L_1 \cap (\Sigma^* \setminus L_2) = L_1 \setminus L_2$

[Again, it's fine to just state the last equality, but you should be able to convince yourself of this and you could write out a formal proof.]

2 (a) Assume L_1 is regular. Since

$\{a\}^*$ is regular, then

$$L_1 \cap \{a\}^* = \{a^n \mid n \geq 1, n \text{ is a perfect cube}\}$$

must be regular (contradicting what was discussed in class and in the handout "Non-Regular...")

(b) Similarly, L_1 is regular, since

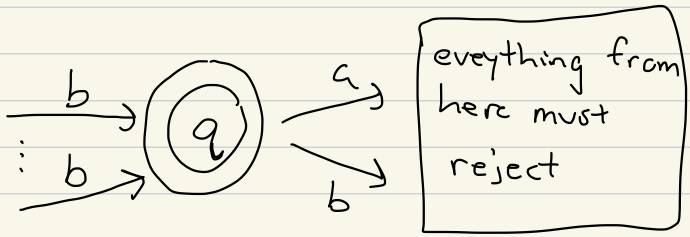
$$L_1 \cap \{a\}^* = \{a^n \mid n \text{ is a perfect cube}\}$$

is non-regular.

(c) If L is regular then

$$L' = L_1 \cap (\{a\}^* b) = \{a^n b \mid n \text{ is a perfect cube}\}$$

is regular; let M be a DFA for L'. Any accepting state, q, of M must look like



i.e.

q has

- one or more incoming edges labelled "b"
- no incoming edges labelled "a"
- all outgoing edges lead to rejecting states.

Hence let M' be obtained from M by

(1) deleting all edges labelled "b"

(an edge labelled "a,b" becomes labelled "a")

and

(2) declaring M' to have the new set of final states, F',

where

$$q \in F' \quad \text{iff} \quad q \text{ has an edge}$$

$$q \xrightarrow{b} \circledcirc \quad \text{in } M$$

[Then $M' = (Q, \{a\}, \delta', q_0, F')$

is a new DFA, where F' is as above,

and δ' is the restriction of δ

to $Q \times \{a\}$, and]

Then for all $s \in \{a\}^*$, we easily see that

$$\left(\begin{array}{l} s \text{ is accepted} \\ \text{by } M' \end{array}\right) \quad \text{iff} \quad \left(\begin{array}{l} sb \text{ is accepted} \\ \text{by } M \end{array}\right)$$

Since M recognizes L' above,

M' recognizes $\{a^n \mid n \text{ is a perfect cube}\}$,

which is impossible.

NOTICE:

This year we did not cover the idea that if $L \subset \{a,b\}^*$ is regular, then so is L where all b's are set to ε.

You can prove this by induction, verifying this for regular expressions of the form $\emptyset$, $\{\varepsilon\}$, $\{\sigma\}$, and then showing that if it holds for R_1, R_2, then it also holds for $(R_1 \cup R_2)$, $(R_1 \circ R_2)$, and (R_1^*).

However, this idea requires Section 1.3

(3) From $M = (Q, \Sigma, \delta, q_0, F)$, create an NFA M' with same alphabet Σ

(1) same states and arrows, with directions reversed

(2) add one more state q_{extra}

(3) declare q_0 to be the only accepting state of M'

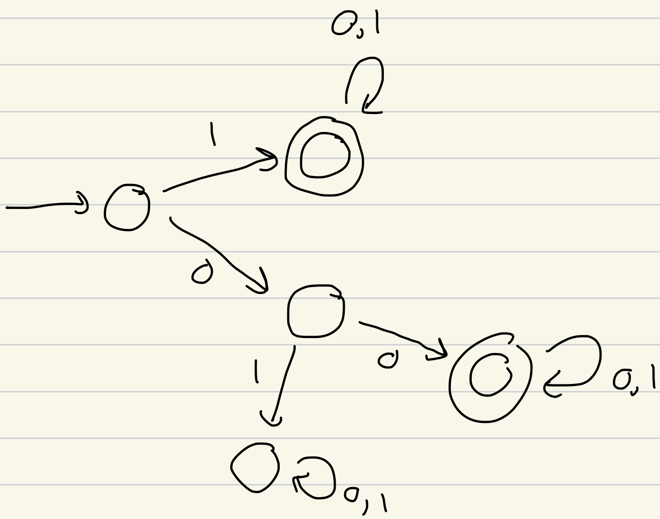
(4) put an ε-transition from q_{extra} to each state $q \in F$, i.e. each accepting state.

Hence $M' = (Q \cup \{q_{extra}\}, \Sigma, \delta', q_{extra}, \{q_0\})$

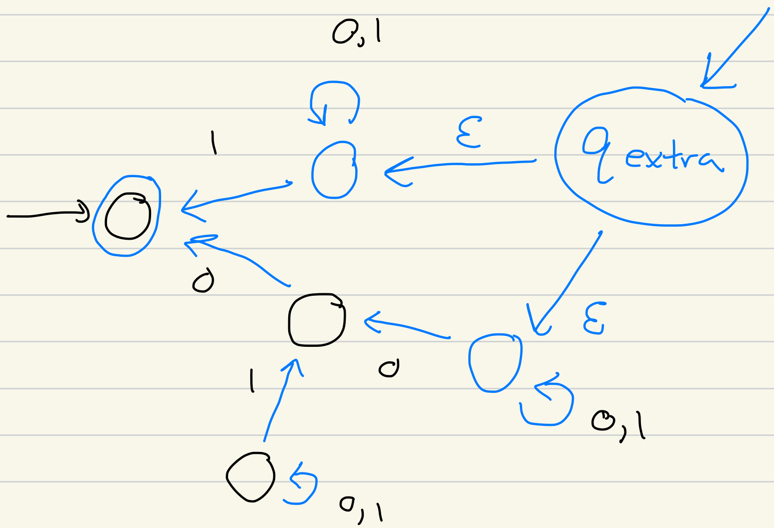
where δ' represents the reversed arrows and the ε-transitions from q_{extra}.

Example: $\Sigma = \{0,1\}$,

$L = \{ w \in \Sigma^* \mid w$ begins with a 1 or with 00 $\}$



L^{rev}:



The reason this works is that if

$w \in L$, then the path through M

can be reversed, yielding a path (one path of possibly many) through M'. Hence

$$w \in L \implies w^{rev} \text{ is accepted by } M'.$$

Conversely if u has a path in M' from q_{extra} to q_0, then the reverse path yields a path in the DFA, beginning at q_0 and ending in an element of F (where $M = (Q, \Sigma, \delta, q_0, F)$) (this path stops at this element of F, since the path through M' begins with a unique ε-transition to an element of F).

Hence

$$u \text{ accepted by } M' \implies u^{rev} \in L \; ;$$

setting $w = u^{rev}$, we equivalently have

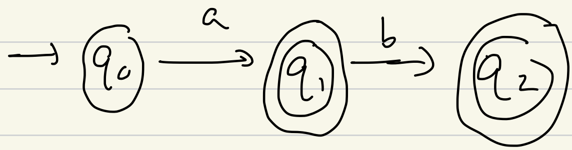
$$w^{rev} \text{ accepted by } M' \Rightarrow w \in L.$$

Hence

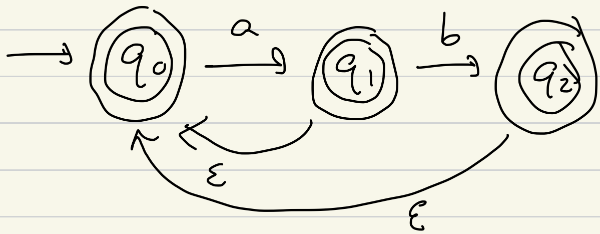
$$w \in L \iff w^{rev} \text{ accepted by } M',$$

and hence M' recognizes L^{rev}.

(4) (a)



(b)



(c) Initial state = all states reachable from q_0 with an ε-transition.

Since there are no ε-arrows from q_0, initial state $= \{q_0\}$

$$\{q_0\} \xrightarrow{a} \{q_0, q_1\}$$

(since $q_0 \xrightarrow{a} q_1$ takes us to q_1, and there is an ε-transition to q_0 from q_0)

$$\{q_0\} \xrightarrow{b} \emptyset$$

(since no arrows labeled b leave q_0)

$\emptyset \xrightarrow{a} \emptyset \qquad \emptyset \xrightarrow{b} \emptyset$ (always)

Similarly

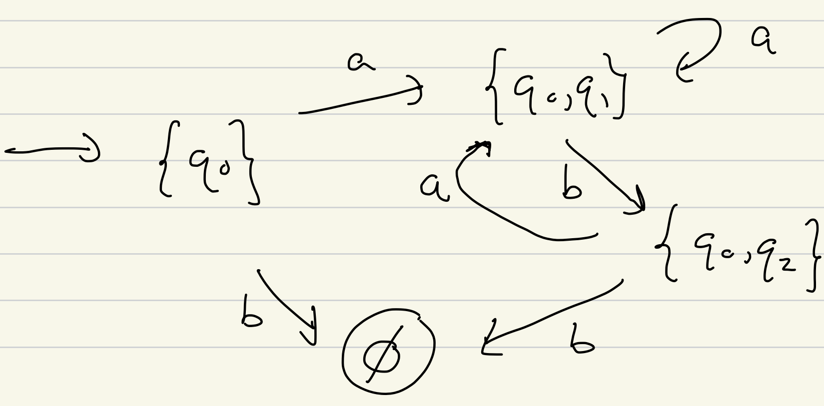
$\{q_0, q_1\} \xrightarrow{a} \{q_0, q_1\}$

$\{q_0, q_1\} \xrightarrow{b} \{q_0, q_2\}$ (only b-arrow from q_0, q_1 takes q_1 to q_2)

$\{q_0, q_2\} \xrightarrow{a} \{q_0, q_1\}$

$\{q_0, q_2\} \xrightarrow{b} \emptyset$

Since no new sets have arisen, we can put this together



Since the final states in (b) are all of q_0, q_1, q_2, the final states are any non-empty set, yielding