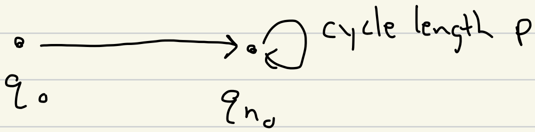
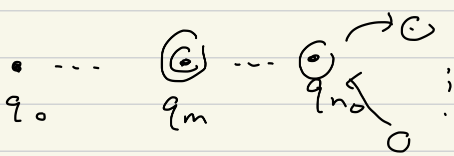
CPSC 421/501 Homework Solutions 5 2023

(1) 6.1.1 on the handout "Non-Regular Languages..."

(a) The DFA must look like



All states in the cycle $q_{n_0}, \ldots, q_{n_0+(p-1)}$ must be rejecting, so q_m must appear before q_{n_0}



so $m \le n_0 - 1$, $p \ge 1$, and

$$\#_{\text{states}} = n_0 + p \ge (m+1)+1 = m+2$$

(b) Similarly all states along the cycle are accepting, and q_m is rejecting, so $m \le n_0 - 1$, $p \ge 1$ so $n_0 + p \ge m+2$.

[Alternatively: to every DFA, M, recognizing L,

by switching the accept and reject states,
we get a DFA recognizing $\Sigma^* \setminus L$.
So for L as in part (b), $\Sigma^* \setminus L$ satisfies
the conditions of part (a); so if a DFA
in part (a) requires at least $m+2$ states,
then the same is true for part (b)]

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(2) 6.1.2

(a) Say that

$$\forall\, n \geq n_0', \qquad a^n \in L \Leftrightarrow a^{n+m'} \in L$$

and

$$\forall\, n \geq n_0, \qquad a^n \in L \Leftrightarrow a^{n+m} \in L$$

then

$\forall\, n \geq \max(n_0', n_0)$

$$a^{n+m} \in L \Leftrightarrow a^n \in L \Leftrightarrow a^{n+m'} \in L$$

so

$$a^{n+m} \in L \iff a^{n+m'} \in L$$

$$\iff a^{n+m+(m'-m)} \in L$$

Setting $k = n+m$, we have

$$a^k \in L \iff a^{k+(m'-m)} \in L$$

for all k s.t. $n = k-m \geq \max(n_0', n)$,

i.e., for $k \geq C$, where $C = m + \max(n_0', n)$.

Hence L is eventually $(m'-m)$-periodic.

(b) Say that L is p'-periodic. We may write $p' = p \cdot r + (p' \bmod p)$ (where $r = \lfloor p'/p \rfloor$)

i.e. $p' = p \cdot r + i$ where $0 \leq i \leq p-1$.

Since L is p'-periodic and p periodic, we have

L is $\begin{cases} (1)\ p'-p \text{ periodic} & (\text{if } p'-p \geq 1), \text{ hence} \\ (2)\ p'-2p \quad '' & ('' \ \ p'-2p \geq 1), \text{ hence} \end{cases}$

(3) $p'-3p$ " (" $p'-3p \geq 1$), hence

⋮

and (by induction on r)

⋮

($p'-rp$) periodic (" $p'-rp \geq 1$).

So if $p' \bmod p = i$ is one of $1, 2, \ldots, p-1$

then L is i-periodic, which is impossible

since $1 \leq i < p$ and p is the periodicity

of L.

(c) If M is a DFA that recognizes L,

and the cycle length of M is p', then

L must be p'-periodic. Hence (b) implies

that p' is divisible by p.

(d) If M looks like

then if

$$n \geq n_0,$$

for all $r > 0$

$$a^n \in L \Leftrightarrow a^{n+rp'} \in L$$

and

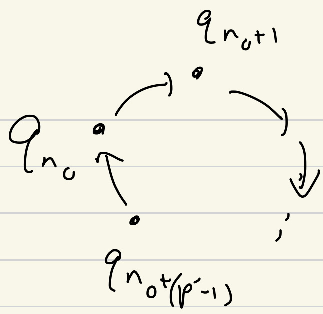
$$a^{n+p} \in L \Leftrightarrow a^{n+p+rp'} \in L$$

Since L is p-periodic, then for r sufficiently large $\quad a^{n+rp'} \in L \Leftrightarrow a^{n+p+rp'} \in L.$

Therefore (for all $n \geq n_0$)

$$a^n \in L \Leftrightarrow a^{n+p} \in L.$$

Hence if $p' > p$, we can replace the cycle of length p' in M



$\leftarrow$ cycle of length p'

by

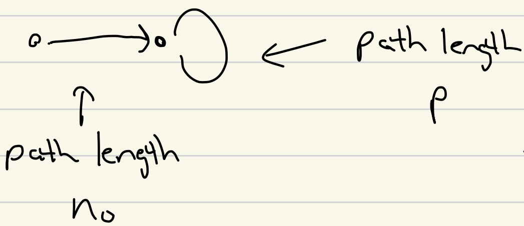
a cycle of length p

The new DFA has n_0+p states, which is fewer than the original DFA (with n_0+p').

6.1.2 (e) If n_0 is the smallest integer with

$$n \geq n_0 \implies a^n \in L \Leftrightarrow a^{n+p} \in L$$

then the same is not true for all $n \geq n_0 - 1$, hence one of a^{n_0-1} and a^{n_0-1+p} is in L, and the other not. By (d), the smallest DFA recognizing L has cycle length p, and the path in L must be of length at least n_0. Hence the number of states is $\geq n_0+p$ in any such DFA, and there is a DFA with n_0+p states (whose shape is



).

Hence the DFA recognizing L with the fewest numbers of states has n_0+p.

(3) 6.1.4 L has period 3, since for large n,

$$a^n \in L \iff (n \bmod 3) = 0.$$

L does not have smaller period, since for all n divisible by 3,

$$a^n \in L \text{ but } a^{n+1}, a^{n+2} \notin L.$$

Since $a^0 \notin L$ and $a^3 \in L$,

it is not true that

$n \geq 0$ implies $a^n \in L \iff a^{n+3} \in L$.

By contrast,

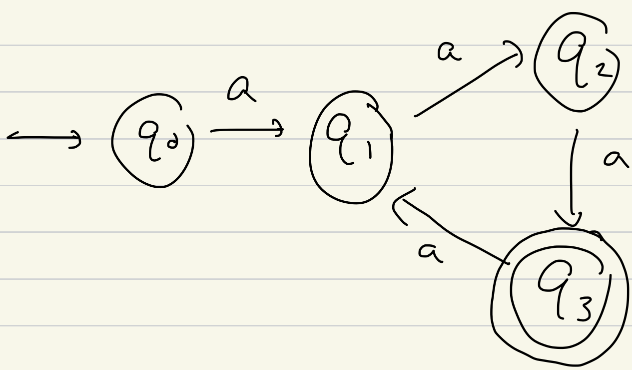
$n \geq 1$ does imply $a^n \in L \iff a^{n+3} \in L$

Since for $n \geq 1$, $a^n \in L$ if $(n \bmod 3) = 0$

and $a^n \notin L$ if $(n \bmod 3) = 1, 2$.

Hence n_0 in 6.1.2 is 1. Hence, by 6.1.2.(e), the minimum numbers of states is $n_0 + p = 4$.

(The DFA is



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(4) 6.1.5

The period of L can be:

1, for example $L = \Sigma^*$

2, for example $L = \{ a^n \mid n \text{ is even} \}$

$2k$ for any $k \in \mathbb{N}$ with $k \geq 2$

for example

$$L = \{ a^n \mid (n \bmod 2k) = 0, 1, 2, 4, 6, \dots, 2k-2 \}$$

since this L is (eventually) $2k$ periodic,

but is not 2 periodic (since $a^n \notin L$ if

$n \bmod (2k) = 3$) and not k periodic

(since $a^n \in L$ if $n \bmod (2k) = 0, 1$

but $a^n \notin L$ if

$$\begin{cases} n \bmod (2k) = k & \text{if } k \text{ is odd} \\ n \bmod (2k) = k+1 & \text{if } k \text{ is even} \end{cases}$$

(If L has period d and $d < 2k$, then d divides $2k$, and therefore (d divides k) or (d divides 2), which are impossible since L is not k-periodic or 2-periodic.)

L cannot have period p' if p' is odd and $p' > 1$, for if so then for large n, $a^n \in L \Leftrightarrow a^{n+p'} \in L$, and for all n odd we have $n+p'$ is even and so $a^{n+p'} \in L$. Hence

$a^n \in L$ for n sufficiently large and odd or even hence L is eventually 1-periodic.

$\boxed{\text{(Finally) } L \text{ can be non-regular, i.e. the period of } L \text{ may not exist,}}$

for example

$$L = \{ a^n \mid n \text{ even or } \underbrace{n = 10^k + 1 \text{ for some } k \in \mathbb{N}}_{\text{(i.e. } n = 11,\ 101,\ 1001,\ 10001, \dots)} \}$$

since this L

(1) does not have period 1 (since n odd and large, such as $n = 10^k + 3$ does not have $a^n \in L$)

(2) does not have an even period p, since

$$10^k + 1 + p \text{ is odd and } < 10^{k+1} + 1$$

for any k with $p < 10^{k+1} - 10^k = 9 \cdot 10^k$

so k with $k > \log_{10}(p/9)$.