(1) 6.1.1 on the handout "Non-Regular Languages..."

(a) The DFA must look like

```
q₀ → q₁ ↪ q₂ ↪ ... ↪ qₙ₀ ↪ q₀
```

Cycle length $p$

All states in the cycle $q₀, ..., qₙ₀+{(p-1)}$ must be rejecting, so $qₘ$ must appear before $q₀, ..., qₙ₀+{(p-1)}$

```
q₀ → q₁ ↪ q₂ ↪ ... ↪ qₙ₀ ↪ q₀
```

So $m ≤ n₀-1$, $p ≥ 1$, and

$$\# \text{ states } = n₀+p ≥ (m+1)+1 = m+2$$

(b) Similarly all states along the cycle are accepting, and $qₘ$ is rejecting, so $m ≤ n₀-1$, $p ≥ 1$ so $n₀+p ≥ m+2$.

[Alternatively: to every DFA, $M$, recognizing $L$,]
by switching the accept and reject states, we get a DFA recognizing $\Sigma^* \setminus L$.

So far $L$ as in part (b), $\Sigma^* \setminus L$ satisfies the conditions of part (a); so if a DFA in part (a) requires at least $m+2$ states, then the same is true for part (b).}

(2) 6.1.2

(a) Say that

\[ \forall n \geq n', \quad a^ne \in L \iff a^{n+m'} \in L \]

and

\[ \forall n \geq n_0, \quad a^ne \in L \iff a^{n+m} \in L \]

then

\[ \forall n \geq \max(n_0', n_0) \]

\[ a^{n+m} \in L \iff a^ne \in L \iff a^{n+m'} \in L \]
So
\[ a^{n+m} \in L \iff a^{n+m'} \in L \]

\[ \iff a^{h+m+(m'-m)} \in L \]

Setting \( k = n+m \), we have

\[ a^k \in L \iff a^{k+(m'-m)} \in L \]

for all \( k \) s.t. \( n = k-m \geq \max(n_0',n) \), i.e., for \( k \geq C' \), where \( C = m+\max(n_0',n) \).

Hence \( L \) is eventually \((m'-m)\)-periodic.

(b) Say that \( L \) is \( p' \)-periodic. We may write \( p' = p \cdot r + (p' \mod p) \) (where \( r = \lfloor p' / p \rfloor \))

i.e., \( p' = p \cdot r + i \) where \( 0 \leq i \leq p-1 \).

Since \( L \) is \( p' \)-periodic and \( p \) periodic, we have

\[ L \text{ is } \begin{cases} (1) \ p' \text{-periodic (if } p' - p \geq 1), \text{ hence } \\ (2) \ p' - 2p \text{ (if } p' - 2p \geq 1), \text{ hence} \end{cases} \]
(3) \( p' - 3p \) \( \equiv (p' - 3p \geq 1) \), hence

and (by induction on \( r \))

\((p' - r\,p) \text{ periodic } \) \( (p' - r\,p \geq 1) \).

So if \( p' \mod p = i \) is one of \( 1, 2, \ldots, p-1 \)

then \( L \) is \( i \)-periodic, which is impossible

since \( 1 \leq i < p \) and \( p \) is the periodicity

of \( L \).

(c) If \( M \) is a DFA that recognizes \( L \),

and the cycle length of \( M \) is \( p' \), then

\( L \) must be \( p' \)-periodic. Hence (b) implies

that \( p' \) is divisible by \( p \).

(d) If \( M \) looks like
then if
\[ n \geq n_0, \]
for all \( r > 0 \)
\[ a^n \in L \iff a^{n+rp'} \in L \]
and
\[ a^{n+p} \in L \iff a^{n+p+rp'} \in L \]

Since \( L \) is \( p \)-periodic, then for \( r \) sufficiently large \( a^{n+rp'} \in L \iff a^{n+p+rp'} \in L \).

Therefore (for all \( n \geq n_0 \))
\[ a^n \in L \iff a^{n+rp} \in L. \]

Hence if \( p' > p \), we can replace the cycle of length \( p' \) in \( M \).
The new DFA has \( n_{0\text{p}} \) states, which is fewer than the original DFA (with \( n_{0\text{p'}} \)).

6.1.2 (e) If \( n_0 \) is the smallest integer with

\[ n \geq n_0 \Rightarrow a^n \in L \iff a^{n+p} \in L \]

then the same is not true for all \( n \geq n_0 - 1 \), hence one of \( a^{n_0 - 1} \) and \( a^{n_0 - 1 + p} \) is in \( L \), and the other not. By (d), the smallest DFA recognizing \( L \) has cycle length \( p \),
and the path in $L$ must be of length at least $n_0$. Hence the number of states is $\geq n_{opt}$ in any such DFA, and there is a DFA with $n_{opt}$ states (whose shape is $\xrightarrow{o} \circ \leftarrow \text{path length} \uparrow p$).

Hence the DFA recognizing $L$ with the fewest number of states has $n_{opt}$.

(3) 6.1.4 $L$ has period 3, since for large $n$, $a^n \in L \iff (n \mod 3) = 0$. $L$ does not have smaller period, since for all $n$ divisible by 3, $a^n \in L$ but $a^{n+1}, a^{n+2} \notin L$. 
Since $a^0 \notin L$ and $a^3 \in L$, it is not true that
\[ n \geq 0 \text{ implies } a^n \in L \iff a^{n+3} \in L. \]

By contrast,
\[ n \geq 1 \text{ does imply } a^n \in L \iff a^{n+3} \in L. \]

Since for $n \geq 1$, $a^n \in L$ if $(n \mod 3) = 0$ and $a^n \notin L$ if $(n \mod 3) = 1, 2$.

Hence $n_0$ in 6.1.2 is 1. Hence, by 6.1.2.(e), the minimum number of states is $n_0 + p = 4$. 

The DFA is
\[
\begin{array}{c}
\text{The DFA} \\
\text{is} \\
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{a} q_0 \rightarrow q_1 \\
\xrightarrow{a} q_2 \\
\xrightarrow{a} q_3 \\
\end{array}
\]
The period of $L$ can be:

1. for example $L = \Sigma^*$
2. for example $L = \{a^n \mid n \text{ is even}\}$

$2k$ for any $k \in \mathbb{N}$ with $k \geq 2$

for example

$$L = \{a^n \mid (n \mod 2k) = 0, 1, 2, 4, 6, \ldots, 2k-2\}$$

since this $L$ is (eventually) $2k$ periodic,

but is not $2$ periodic (since $a^k \notin L$ if $n \mod (2k) = 3$) and not $k$ periodic

(since $a^n \in L$ if $n \mod (2k) = 0, 1$ but $a^n \notin L$ if $n \mod (2k) = 3$)
\[
\begin{cases}
  \text{if } k \text{ is odd} & n \mod (2k) = k \\
  \text{if } k \text{ is even} & n \mod (2k) = k + 1
\end{cases}
\]

(If \( L \) has period \( d \) and \( d \leq 2k \),
then \( d \) divides \( 2k \), and therefore
\((d \text{ divides } k) \) or \((d \text{ divides } 2)\), which
are impossible since \( L \) is not \( k \)-periodic
or \( 2 \)-periodic.)

\[ L \text{ cannot have period } p' \text{ if } p' \text{ is odd and } p' \geq 1, \text{ for if so then for large } n, \ A^n \in L \iff A^{n+p'} \in L, \text{ and for all } n \text{ odd we have } n+p' \text{ is even and so } A^{n+p'} \in L. \text{ Hence }

A^n \in L \text{ for } n \text{ sufficiently large and odd or even} \]
hence \( L \) is eventually \( 1 \)-periodic.

(Finally) \( L \) can be non-regular, i.e. the period of \( L \) may not exist, for example

\[
L = \left\{ a^n \mid \text{n even or } n = 10^k + 1 \text{ for some } k \in \mathbb{N} \right\}
\]

(i.e. \( n = 11, 101, 1001, 10001, \ldots \))

since this \( L \)

1. does not have period \( 1 \) (since \( n \) odd and large, such as \( n = 10^k + 3 \) does not have \( a^n \in L \))
2. does not have an even period \( p \), since

\[
10^k + 1 + p \text{ is odd and } < 10^{k+1} + 1
\]

for any \( k \) with \( p < 10^{k+1} - 10^k = 9 \cdot 10^k \)

so \( k \) with \( k > \log_{10} (p/9) \).