CPSC 421/501 Fall 2023 Homework Solutions 2

[Note: remarks in brackets, [ ], are optional]

7.2.2 Let $f(1)=f(2)=f(3)=\{1,2,3\}$. Then for all $s=1,2,3$, $s\in\{1,2,3\}=f(s)$. Hence

$$T=\{s\in\{1,2,3\}\mid s\in f(s)\}=\{1,2,3\},$$

and $\{1,2,3\}$ lies in the image of f.

[There are many other examples, e.g. $f(1)=f(2)=f(3)=\emptyset$ ...]

7.2.3 (a) Since $1\in\{1,2\}=f(1)$, [it is not true that $1\notin f(1)$ and hence] $1\notin T$. Since $1\in f(1)$ and $1\notin T$, $T\neq f(1)$

(b) If $f(2)=\emptyset$, then $2\notin\emptyset=f(2)$ and so $2\in T$. On the other hand, if $f(2)=\{1,2,3\}$, then $2\in\{1,2,3\}=f(2)$, so $2\notin T$. Hence both $2\in T$ and $2\notin T$ are possible.

7.2.4.

$$n \in f(n) \iff n + \frac{n}{2} = k^2 \text{ for some } k \in \mathbb{N}$$

$$\iff \frac{3n}{2} = k^2 \text{ for some } k \in \mathbb{N}$$

$$\iff n = \frac{2k^2}{3} \text{ for some } k \in \mathbb{N}$$

Since $2k^2/3$ is an integer, the prime factorization of k^2 must include 3 as a factor, and hence so does k. Hence $k = 3a$ for some $a \in \mathbb{N}$, and hence $n = 6a^2$.

So $n \in f(n) \iff n \in \{6a^2 \mid a \in \mathbb{N}\} = \{6, 6\cdot 4, 6\cdot 9, \ldots\}$,

So $n \notin f(n) \iff n \in \mathbb{N} \setminus \{6, 6\cdot 4, 6\cdot 9, \ldots\}$

So

$$T = \mathbb{N} \setminus \{6, 6\cdot 4, 6\cdot 9, \ldots\}$$

You only need one answer that looks something like this...

$$= \mathbb{N} \setminus \{6a^2 \mid a \in \mathbb{N}\}$$

$$= \{n \in \mathbb{N} \mid n \text{ is not } 6 \text{ times a perfect square}\}$$

7.2.9

Either $t \in T$ or $t \notin T$.

<u>If $t \in T$</u> $= \{ s \in S \mid t \notin f(g(t)) \}$ then

$t \notin f(g(t))$. But $g(t) = t'$, so $t \notin f(t')$.

But $f(t') = T$, so $t \notin T$. This contradicts

the assumption that $t \in T$.

<u>If $t \notin T$</u>, then similarly t does not satisfy

$t \notin f(g(t))$, so $t \in f(g(t)) = f(t') = T$, so

$t \in T$, contradicting the assumption that

$t \notin T$.

7.2.13. (a) $\forall r \in R$, r loves themself, so $r \notin T$. Hence $T = \emptyset = \{\}$.

(b) If David loves David (themself), then David $\notin T$. But also

$$\text{David} \in \{\text{people whom David loves}\}$$

So

$$T \neq \{\text{people whom David loves}\}.$$