CPSC 421/501 Homework Solutions 10

(1) (a) By distributivity of $\wedge$ and $\vee$,

$$\left(\bigwedge_{i=1}^{3} u_i\right) \vee \left(\bigwedge_{j=1}^{3} v_j\right) \vee \left(\bigwedge_{k=1}^{3} w_i\right)$$

$$= \bigwedge_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}} \left(u_i \vee v_j \vee w_k\right)$$

Size = # variables: We have 27 clauses, each clause with 3 literals, so size = $27 \cdot 3 = 81$

(b) By distributivity we have

$$\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee \left(\bigwedge_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}} \left(u_i \vee v_j \vee w_k\right)\right)$$

$$= \bigwedge_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}} \left(\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee u_i \vee v_j \vee w_k\right)$$

with 27 clauses of 6 literals per clause,

hence size $= 27 \cdot 6 = 162$

(c) We replace each clause

$$\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee u_i \vee v_j \vee w_k$$

by
$$\begin{aligned}
&(\neg x_1 \vee \neg x_2 \vee z_{1,ijk}) \wedge \\
&(\neg z_{1,ijk} \vee \neg x_3 \vee z_{2,ijk}) \wedge \\
&(\neg z_{2,ijk} \vee u_i \vee z_{3,ijk}) \wedge \\
&(\neg z_{3,ijk} \vee v_j \vee w_k)
\end{aligned}$$

therefore getting

$$\bigwedge_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}} \left[\begin{array}{l} (\neg x_1 \vee \neg x_2 \vee z_{1,ijk}) \wedge \\ (\neg z_{1,ijk} \vee \neg x_3 \vee z_{2,ijk}) \wedge \\ (\neg z_{2,ijk} \vee u_i \vee z_{3,ijk}) \wedge \\ (\neg z_{3,ijk} \vee v_j \vee w_k) \end{array} \right]$$

Which is a 3CNF with $27 \cdot 4$ clauses,

hence of size $3 \cdot 27 \cdot 4 = 81 \cdot 4 = 324$

(d) The values of $z_{1,ijk}$ and $z_{2,ijk}$ depend only on x_1, x_2, x_3, which are independent of i, j, k; for example, if $\neg x_1 = \neg x_2 = \neg x_3 = F$, then we must take $z_{1ijk} = T$ and $z_{2ijk} = T$; by constrast, if $\neg x_1$ or $\neg x_2 = T$, then we can take $z_{1ijk} = z_{2ijk} = z_{3ijk} = F$, or if $\neg x_3 = T$ we can take $z_{1ijk} = T$ and $z_{2ijk} = z_{3ijk} = F$

Hence the above formula can be shortened to

$$(\neg x_1 \vee \neg x_2 \vee z_1) \wedge (\neg z_1 \vee \neg x_3 \vee z_2)$$

$$\wedge \bigwedge_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}} \left[(\neg z_2 \vee u_i \vee z_{3,ijk}) \wedge (\neg z_{3,ijk} \vee v_j \vee w_k) \right].$$

which is of size $6 + 27 \cdot 6 = 28 \cdot 6 = 168$.

Similarly the value of $z_{3,ijk}$ we need to take depends only on $\neg z_2$ and u_i. Hence we can write the above as

$$(\neg x_1 \vee \neg x_2 \vee z_1) \wedge (\neg z_1 \vee \neg x_3 \vee z_2) \wedge$$

$$\bigwedge_{i=1,2,3} \left[(\neg z_2 \vee u_i \vee z_{3,i}) \wedge \bigwedge_{\substack{j=1,2,3\\ k=1,2,3}} (\neg z_{3,i} \vee v_j \vee w_k) \right]$$

which is of size $6 + 3(3+27) = 96$

(2) (a) By distributing $\wedge$ and $\vee$, this formula becomes a 3CNF of size 324

(b) If $y_{ij} = T$ with $j=1$, then a L at step i makes $y_{i+1,j} = y_{i+1,1} = L$, rather than $y_{i+1,j-1} = L$ ($y_{i+1,j-1} = y_{i+1,0}$ does not exist). So the $y_{i+1,j-1}$ is replaced with $y_{i+1,j}$ (or $y_{i+1,1}$); everything else is the same. Hence the 3CNF is the same size.

(c) With 5 transitions we get 3^5 clauses, each clause being at OR of 8 literals (instead of 6). Hence we get an 8 CNF with $3^5 = 243$ clauses. The analogous trick is to write

$$a_1 \vee a_2 \vee \cdots \vee a_8 \quad \text{is true}$$

$$\iff$$

$$(a_1 \vee a_2 \vee z_1) \wedge (\neg z_1 \vee a_3 \vee z_2) \wedge (\neg z_2 \vee a_4 \vee z_3)$$
$$(\neg z_3 \vee a_5 \vee z_4) \wedge (\neg z_4 \vee a_6 \vee z_5) \wedge (\neg z_5 \vee a_7 \vee a_8)$$

is satisfiable. Hence we can convert the 8CNF with 3^5 clauses into a 3 CNF of $3^5 \cdot 6$ clauses, which is of size $3^6 \cdot 6 = 729 \cdot 6 = 4374$

(3) No: the reduction, as described takes time 2^n times some polynomial in n (to evaluate $f(a_1, \ldots, a_n)$). Hence the reduction is not polynomial time.

(4) Say that the reduction from A to B computes a function f.

On input w to A of length n, the reduction $A \leq_p B$ runs in time $5n^4$; hence $f(w)$ is of length at most $5n^4$ hence combining this with the reduction $B \leq_p C$ runs in time $3N^8$ where $N = 5n^4$, i.e. in time $3(5n^4)^8 = 3 \cdot 5^8 \cdot n^{32}$.

(5)(a)

If C_i contains ~~the~~ a variable and its negation, for example $\neg x_3 \vee x_3 \vee x_4$, then this clause evaluates to T regardless of the values of $x_1, \ldots, x_n$. Otherwise l_{i1}, l_{i2}, l_{i3} involve 1 to 3 literals that are not negations of one another.

So if l_{i1}, l_{i2}, l_{i3} involve 3 distinct literals, we have $l_{i1} = x_a$ or $\neg x_a$, $l_{i2} = x_b$ or $\neg x_b$, and $l_{i3} = x_c$ or $\neg x_c$, with a, b, c distinct.

In this case $l_{i1} \vee l_{i2} \vee l_{i3}$ equals F iff $l_{i1} = l_{i2} = l_{i3} = F$, which is possible when x_a, x_b, x_c are chosen appropriately. In this case the other $n-3$ variables can be set to either T or F. Hence there are <u>exactly</u> $2^{n-3} = 2^n/8$ ways to set $x_1, \ldots, x_n$ to T, F to make

$$l_{i1} \vee l_{i2} \vee l_{i3} = F.$$

Similarly there are exactly 2^{n-2} or 2^{n-1} ways when some of l_{i1}, l_{i2}, l_{i3} are repeated.

(5b) If $l_{i1} \vee l_{i2} \vee l_{i3}$ evaluates to F, then

$$\bigwedge_{i=1}^{m} l_{i1} \vee l_{i2} \vee l_{i3} \text{ evaluates to F.}$$

Hence if $f = f(x_1, \ldots, x_n)$ equals a 3CNF formula, then either

(1) each clause, C_i, evaluates to T regardless

or $x_1, \ldots, x_n$, so $f(x_1, \ldots, x_n) = T$

for all $x_1, \ldots, x_n$,

OR

(2) Some c_i evaluates to F

for $2^{n-3} = 2^n/8$ possible

values of $x_1, \ldots x_n$. Hence $c_1 \wedge c_2 \wedge \ldots \wedge c_m$

evaluates to F on at least $2^n/8$ of its values.

(5c) $f(x_1, \ldots, x_4) = x_1 \vee x_2 \vee x_3 \vee x_4$

equals F only when $(x_1, x_2, x_3, x_4) = (F, F, F, F)$

i.e. on $\frac{1}{16} 2^n$ values. Since f is not

identically T, but f is F on only $\frac{1}{16} 2^n$

of its values, f cannot be written as a

3CNF formula.