(1) (a) By distributivity of \( \land \) and \( \lor \),

\[
(\bigwedge_{i=1}^{3} u_i) \lor (\bigwedge_{j=1}^{3} v_j) \lor (\bigwedge_{k=1}^{3} w_k)
\]

\[= \bigwedge_{i=1,2,3} (u_i \lor v_j \lor w_k)\]

Size = \# variables: We have 27 clauses, each clause with 3 literals, so size = 27 \cdot 3 = 81

(b) By distributivity we have

\[
\neg x_1 \lor \neg x_2 \lor \neg x_3 \lor \left(\bigwedge_{i=1,2,3} (u_i \lor v_j \lor w_k)\right)
\]
\[
\bigwedge_{i=1,2,3} \bigwedge_{j=1,2,3} \bigwedge_{k=1,2,3} (x_i \lor \neg x_2 \lor \neg x_3 \lor u_i \lor v_j \lor w_k)
\]

with 27 clauses of 6 literals per clause, hence size = 27 \times 6 = 162

(c) We replace each clause

\[x_i \lor \neg x_2 \lor \neg x_3 \lor u_i \lor v_j \lor w_k\]

by

\[\neg x_1 \lor \neg x_2 \lor z_{1,ijk} \land \]

\[\neg z_{1,ijk} \lor \neg x_3 \lor z_{2,ijk} \land \]

\[\neg z_{2,ijk} \lor u_i \lor z_{3,ijk} \land \]

\[\neg z_{3,ijk} \lor v_j \lor w_k\]

therefore getting
\[ \bigwedge_{i=1,2,3} \left( \neg x_i v \neg x_2 v \neg x_3 v Z_1,ijk \right) \wedge \]
\[ \left( \neg Z_1,ijk v \neg x_3 v Z_2,ijk \right) \wedge \]
\[ \left( \neg Z_2,ijk v u; v Z_3,ijk \right) \wedge \]
\[ \neg Z_3,ijk v vj v w_k \]

which is a 3 CNF with 27·4 clauses, hence of size 3·27·4 = 81·4 = 324.

(d) The values of \( Z_1,ijk \) and \( Z_2,ijk \) depend only on \( x_1, x_2, x_3 \), which are independent of \( i,j,k \); for example, if \( \neg x_1 = \neg x_2 = \neg x_3 = F \), then we must take \( Z_1,ijk = T \) and \( Z_2,ijk = T \); by contrast, if \( \neg x_1 \) or \( \neg x_2 = T \), then we can take \( Z_1,ijk = Z_2,ijk = Z_3,ijk = F \), or if \( \neg x_3 = T \) we can take \( Z_1,ijk = T \) and \( Z_2,ijk = T, Z_3,ijk = F \).
Hence the above formula can be shortened to

\[ (\neg x_1 \lor \neg x_2 \lor \neg z_1) \land (\neg z_1 \lor x_3 \lor z_2) \land \left( \bigwedge_{i=1,2,3} \left( \neg z_2 \lor u_i \lor z_3,ijk \right) \land \left( \bigwedge_{i=1,2,3} \left( \neg z_3,ijk \lor v_j \lor w_k \right) \right) \right) \]

which is of size $6 + 27 \cdot 6 = 28 - 6 = 168$.

Similarly the value of $z_{3,ijk}$ we need to take depends only on $\neg z_2$ and $u_i$. Hence we can write the above as

\[ (\neg x_1 \lor \neg x_2 \lor \neg z_1) \land (\neg z_1 \lor x_3 \lor z_2) \land \left( \bigwedge_{i=1,2,3} \left( \neg z_2 \lor u_i \lor z_3,ijk \right) \land \left( \bigwedge_{i=1,2,3} \left( \neg z_3,ijk \lor v_j \lor w_k \right) \right) \right) \]

which is of size $6 + 3 \left( 3 + 27 \right) = 96$. 
(2) (a) By distributing $\land$ and $\lor$, this formula becomes a 3CNF of size 324.

(b) If $Y_{ij} = T$ with $j=1$, then a $L$ at step $i$ makes $Y_{i+1,j} = Y_{i+1,1} = L$, rather than $Y_{i+1,j-1} = L$ ($Y_{i+1,j-1} = Y_{i+1,0}$ does not exist). So the $Y_{i+1,j-1}$ is replaced with $Y_{i+1,j}$ (or $Y_{i+1,1}$). Everything else is the same. Hence the 3CNF is the same size.

(c) With 5 transitions we get $3^5$ clauses, each clause being at OR.
of 8 literals (instead of 6). Hence we get an 8-CNF with $3^5 = 243$ clauses. The analogous trick is to write

$$ a_1 \lor a_2 \lor \cdots \lor a_8 \text{ is true} $$

(\Rightarrow)

$$( a_1 \lor a_2 \lor \overline{z}_1 ) \land ( \overline{z}_1 \lor a_3 \lor \overline{z}_2 ) \land ( \overline{z}_2 \lor a_4 \lor \overline{z}_3 )$$

$$( \overline{z}_3 \lor a_5 \lor \overline{z}_4 ) \land ( \overline{z}_4 \lor a_6 \lor \overline{z}_5 ) \land ( \overline{z}_5 \lor a_7 \lor a_8 )$$

is satisfiable. Hence we can convert the 8-CNF with $3^5$ clauses into a 3-CNF of $3^5 \cdot 6$ clauses, which is of size $3^6 \cdot 6 = 729 \cdot 6 = 4374$
(3) No: the reduction, as described takes time \( 2^n \) times some polynomial in \( n \) (to evaluate \( f(a_1, \ldots, a_n) \)). Hence the reduction is not polynomial time.

(4) Say that the reduction from \( A \) to \( B \) computes a function \( f \).

On input \( w \) to \( A \) of length \( n \), the reduction \( A \leq_p B \) runs in time \( 5n^4 \); hence \( f(w) \) is of length at most \( 5n^4 \) hence combining this with the reduction \( B \leq_p C \) runs in time \( 3N^8 \) where \( N = 5n^4 \), i.e. in time \( 3(5n^4)^8 = 3.58 \cdot n^{32} \).
If \( C_i \) contains the a variable and its negation, for example \( \neg x_3 v x_3 v x_4 \), then this clause evaluates to \( T \) regardless of the values of \( x_1, \ldots, x_n \). Otherwise \( l_{i_1}, l_{i_2}, l_{i_3} \) involve 1 to 3 literals that are not negations of one another. So if \( l_{i_1}, l_{i_2}, l_{i_3} \) involve 3 distinct literals, we have \( l_{i_1} = x_a \lor \neg x_a \), \( l_{i_2} = x_b \lor \neg x_b \), and \( l_{i_3} = x_c \lor \neg x_c \), with \( a, b, c \) distinct. In this case \( l_{i_1} v l_{i_2} v l_{i_3} \) equals \( F \) iff \( l_{i_1} \equiv l_{i_2} \equiv l_{i_3} = F \), which is
possible when $X_a, X_b, X_c$ are chosen appropriately. In this case the other $n-3$ variables can be set to either $T$ or $F$. Hence there are exactly $2^{n-3} = 2^{n/8}$ ways to set $X_1, \ldots, X_n$ to $T, F$ to make
\[ l_1 \lor l_2 \lor l_3 = F. \]
Similarly there are exactly $2^{n-2}$ or $2^{n-1}$ ways when some of $l_1, l_2, l_3$ are repeated.

(5b) If $l_1 \lor l_2 \lor l_3$ evaluates to $F$, then
\[ \bigwedge_{i=1}^m l_1 \lor l_2 \lor l_3 \text{ evaluates to } F. \]
Hence if $f=f(x_1, \ldots, x_n)$ equals a 3CNF formula, then either
0 each clause, $c_i$, evaluates to $T$ regardless
or \( x_1, \ldots, x_n \), so \( f(x_1, \ldots, x_n) = T \)

for all \( x_1, \ldots, x_n \).

OR

2) Some \( c_i \) evaluates to \( F \)

for \( 2^{n-3} = 2^n/8 \) possible

values of \( x_1, \ldots, x_n \). Hence \( c_1, c_2, \ldots, c_m \)

evaluates to \( F \) on at least \( 2^n/8 \) of its values.

\((5c)\) \( f(x_1, \ldots, x_n) = x_1 \lor x_2 \lor x_3 \lor x_4 \)

equals \( F \) only when \( (x_1, x_2, x_3, x_4) = (F, F, F, F) \)

i.e. on \( \frac{1}{16} 2^n \) values. Since \( f \) is not

identically \( T \), but \( f \) is \( F \) on only \( \frac{1}{16} 2^n \)

of its values, \( f \) cannot be written as a

3CNF formula.