GROUP HOMEWORK 11, CPSC 421/501, FALL 2023

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**

For the problems below, $\wedge, \vee, \neg$ denote the logical operations AND,OR,NOT respectively.

For the problems below, recall that to show that L is NP-complete one needs to show (i) $L \in$ NP, and (ii) any language in NP can be reduced to L (with a polynomial time reduction); (ii) is equivalent to reducing some known NP-complete language to L (with a polynomial time reduction).

(0) Who are your group members? Please print if writing by hand.

(1) Let PARTITION be the language of descriptions of sequences of positive integers, $x_1, \ldots, x_N$ such that for some $I \subset \{1, \ldots, N\}$ we have

$$\sum_{i \in I} x_i = \sum_{j \notin I} x_j$$

(i.e., the sequence $x_1, \ldots, x_N$ can be partitioned into two subsequences whose sums are equal. Like SUBSET-SUM, we view partition as a language over $\Sigma = \{0, 1, \ldots, 9, \#\}$.[1]

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[1] Recall our notation in class: the problem of finding a subset of the sequence (or multiset) $12, 12, 13, 14$ that sums to 24 as 12#12#13#14#24, where the # acts as a separator and the last number, 24, is the desired sum of the sequence (or multiset) $12, 12, 13, 14$; since $12 + 12 = 24$, the string 12#12#13#14#24 lies in SUBSET-SUM.

(a) Show that if $\langle x_1,\dots,x_N,t\rangle \in$ SUBSET-SUM,
$$\langle x_1,\dots,x_N,y-t,t\rangle \in \text{PARTITION}$$
if $y-t>0$.

(b) Show that the map
$$\langle x_1,\dots,x_N,t\rangle \mapsto \langle x_1,\dots,x_N,x_1+\cdots+x_N-t,t\rangle$$
is NOT a reduction of SUBSET-SUM to PARTITION. More specifically, give an example of $\langle x_1,\dots,x_N,t\rangle$ that is not in SUBSET-SUM, such that $\langle x_1,\dots,x_N,x_1+\cdots+x_N-t,t\rangle$ is in PARTITION.

(c) Show that given $\langle x_1,\dots,x_N,t\rangle$, one can find an integer B such that (i) B is of size polynomial in $\langle x_1,\dots,x_N,t\rangle$, and (ii) $\langle x_1,\dots,x_N,t\rangle \in$ SUBSET-SUM iff $\langle x_1,\dots,x_N,B+x_1+\cdots+x_N-t,t+B\rangle$ is in PARTITION.

(d) Using the previous part, show that PARTITION is NP-complete.

(2) Using the hints in [Sip], Problem 7.29, page 325, show that 3COLOR (the language of descriptions of graphs that are 3-colourable) is NP-complete.

(3) Given the fact that 3COLOR is NP-COMPLETE, show that 4COLOR, the set of descriptions of graphs that have a 4-colouring, is NP-complete.

(4) Recall that $\text{Threshold}_{k,n}$ refers to the function $\{T,F\}^n \to \{T,F\}$ given by
$$\text{Threshold}_{k,n}(x_1,\dots,x_n) \stackrel{\text{def}}{=} \begin{cases} T & \text{at least } k \text{ of } x_1,\dots,x_n \text{ equal } T, \text{ and} \\ F & \text{otherwise.} \end{cases}$$
For $n\in\mathbb{N}$, let L_n denote the minimum size of a formula for $\text{Threshold}_{2,n}$.

(a) Explain why if $1\le m\le n$, $\text{Threshold}_{2,n}(x_1,\dots,x_n)$ equals
$$\text{Threshold}_{2,m}(x_1,\dots,x_m) \vee \text{Threshold}_{2,n-m}(x_{m+1},\dots,x_n) \vee \Bigl((x_1\vee\ldots\vee x_m) \wedge (x_{m+1}\vee\ldots\vee x_n) \Bigr)$$

(b) Using part (a), explain why for n even, we have $L_n \le 2L_{n/2}+n$.

(c) Using part (b), conclude that if $n=2^k$ for some $k\in\mathbb{N}$, then $L_n \le n\log_2 n$.

(d) Recall that in class, for $n=2^k$ we gave a different formula for $\text{Threshold}_{2,n}$, using k clauses $c_1,\dots,c_k$, where each c_i divides the variables $x_1,\dots,x_n$ into two groups based on the i-th bit in the binary (base 2) expansions of $1,\dots,n$. How is this related to the formula for $\text{Threshold}_{2,n}$ given in parts (a,b,c)?

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