GROUP HOMEWORK 8, CPSC 421/501, FALL 2023

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Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) You do not have to use LaTeX for homework, but **homework that is too difficult to read will not be graded.**
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**

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(0) Who are your group members? Please print if writing by hand.

(1) Let
$$L = \{\langle M \rangle \mid M \text{ is a T.m. that halts on input } \epsilon\}$$
(where ϵ is the empty string). Show that L is (Turing) undecidable but (Turing) recognizable. What can you say about the complement of L? [You can use your answer, or part of your answer, to Problem (1) if you like; you don't need to repeat the entire part twice.]

(2) Let
$$L = \{\langle M, q \rangle \mid M \text{ is T.m. that reaches state } q \text{ on at least one input}\}$$
i.e., on some input, i, the Turing machine M reaches a configuration which is at state q. Show that L is (Turing) undecidable but (Turing) recognizable. What can you say about the complement of L?

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(3) Let $G = (V, E)$ be an undirected graph (see page 10, Chapter 0 of [Sip] or class notes Nov. 6,8) where $V = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$.

[Recall that if G is a graph on vertex set $\{1, \ldots, N\}$, then the description of G, $\langle G \rangle$, consists of specifying N (in base 10), followed by a list of pairs $i, j \in \{1, \ldots, N\}$ such that there is an edge $\{i, j\} \in E$. Hence the alphabet for such a description is $\{0, 1, \ldots, 9, \#\}$]. Hence the graph on vertex set $\{1, \ldots, 20\}$ with three edges $\{3, 15\}, \{16, 4\}, \{20, 10\}$ has description $20\#3\#15\#16\#4\#20\#10$.

[The point of this exercise is to point out that $\langle G \rangle$ can be very different from $|V|$ and $|E|$; for example, if G has N vertices and no edges, then $\langle G \rangle$ will be of size $O(\log N)$ with the conventions above.]

We say that G contains a *half-sized clique* if G has a subset of vertices $V' \subset V$ such that $|V'| \geq |V|/2$, and every pair of vertices in V' has an edge between them. We let

$$\text{HALF-CLIQUE} = \bigl\{ \langle G \rangle \bigm| G \text{ is a graph that contains a half-sized clique} \bigr\}.$$

(a) Show that if $G = (V, E)$ is a graph where $V = \{1, \ldots, N\}$ and where $n = |\langle G \rangle|$, then $|E| \leq n/4$.[1]

(b) Show that if G is a graph where $V = \{1, \ldots, N\}$ where with $\langle G \rangle$ of length less than $\binom{N/2}{2} = N(N-2)/8$, then $G \notin$ HALF-CLIQUE. [You can improve this to $N(N-2)/2$ using part (a).]

(c) Give an algorithm to decide HALF-CLIQUE in time $p(n)2^{n/4}$, where $p(n)$ is a polynomials, where, as usual, n is the length of the input, i.e., of $\langle G \rangle$.

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[1] One can improve this bound on $|E| \leq O(n/\log n)$ with some extra work.