A Brief Tour of Term Rewriting

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Term Rewriting Systems

- **What are Term Rewriting Systems?**
  - A model for transforming *terms* via application of *rewrite rules*
  - Rules define structural transformations
- **Why are Term Rewriting Systems Useful?**
  - Can serve as a nondeterministic model of computation
  - Expressive, but with very simple syntax and semantics
  - Admit interesting properties (today: Termination and Confluence)

Today

1. Introduce Term Rewriting Systems
2. Describe Properties of Term Rewriting Systems
   1. Confluence
   2. Termination
3. Prove Termination of Term Rewriting Systems

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Term Rewriting Systems, Formally

From (Klop, 1990)

We consider terms built from an alphabet $\Sigma = (F, V)$ such that:

- F is a set of *function symbols*, each associated with an arity
  - The arity of a function is the number of arguments it is supposed to have
  - We denote the arity of a function f as $\alpha(f)$
- V is a countably infinite set of *variables* (typically denoted $x, x_1, y, y_1, \ldots$)

Term Rewriting Systems, Formally

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Terms over Σ are denoted as $T(\Sigma)$, where:

1. $x \in V \implies x \in T(\Sigma)$
2. $f \in F \land \alpha(f) = n \land t_1, \ldots, t_n \in T(\Sigma) \implies f(t_1, \ldots, t_n) \in T(\Sigma)$

To make things look nice:

- If $\alpha(f) = 0$, we can write $f()$ as f
- If $\alpha(f) = 2$, we can write $f(t, u)$ as $t\ f\ u$

Example: Addition and Peano Numbers

$F = \{z, s, + \}$

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$\ldots$

Example: Addition and Peano Numbers

$F = \{z, s, +\}$ $z + z$

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Example: Addition and Peano Numbers

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z	$z + s(z)$
$s(z)$	$s(z) + s(z)$
$s(s(z))$	$(z + z) + s(z)$
$\ldots$	$\ldots$

Term Rewriting Systems, Informally

From (Klop, 1990)

A *rewrite rule* r is a pair of terms (t, u), denoted $r : t \to u$ or simply $t \to u$

A rewrite rule $r : t \to u$ defines a binary relation $\to_r$ on T, informally:

- $t' \to_r u'$ if t "matches" some subterm s of t' via a substitution σ
- u' is the result of replacing s in t' with substitution σ applied to u'

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- Rewrite Rule $r : x + z \to x$

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- Rewrite Rule $r : x + z \to x$
- Application $(s(z) + z)+s(z)$

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- Application $(s(z) + z) + s(z) \rightarrow_r s(z) + s(z)$

Term Rewriting Systems, Formally

From (Klop, 1990)

A *rewrite rule* r is a pair of terms (t, u), denoted $r : t \to u$ or simply $t \to u$

A *substitution* σ is a mapping $V \to T$

- $\sigma(t)$ denotes the replacement of each variable $v \in t$ with $\sigma(v)$

A *context* C is a term with a single "hole", e.g. $f(x, \square)$ or the trivial context $\square$

- $C[t]$ represents the term obtained by filling the hole with t, e.g $f(x, t)$ or t

A rewrite rule $r : t \to u$ defines a binary relation $\to_r$ where:

$C[\sigma(t)] \to_r C[\sigma(u)]$ for all contexts C, substitutions σ

Example

Klop

Rules:

$$r_1 : x + z \to x$$

$$r_2 : x + s(y) \to s(x + y)$$

Rewriting:

$$s(z) + s(s(z))$$

Example

Klop

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$r_1 : x + z \to x$

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$\sigma = \{x \to s(z), y \to s(z)\}$

$C = \square$

Rewriting:

$s(z) + s(s(z)) \to_{r_2}$

$s(s(z) + s(z))$

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Rewriting:

$s(z) + s(s(z)) \to_{r_2}$

$s(s(z) + s(z)) \to_{r_2}$

$s(s(s(z) + z)) \to_{r_1}$

$s(s(s(z)))$

Term Rewriting Systems, Finally

From (Klop, 1990)

A Term Rewriting System is the pair (Σ, R) where:

- Σ is an alphabet, and
- R is a set of rewrite rules over $T(\Sigma)$

Typically, the rewrite rules denote axioms for some theory

A TRS defines a binary relation $\rightarrow_R$ on $T(\Sigma)$, such that:

- $t \rightarrow_R u$ iff u can be obtained from t by applying a rule from R

Reasoning Enabled by TRS

A relation of interest is $\rightarrow_R^*$, the reflexive transitive closure of $\rightarrow_R$

Intuition: $t \rightarrow_R^* u$ iff u can be obtained from t by applying zero or more rewrite rules from R

A common application: compute the set of terms obtained via rewriting from t, namely the set: $\{u \in T \mid t \rightarrow_R^* u\}$

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Also of interest: obtaining the *normal form* of t.

Find a u such that $t \rightarrow_R^* u \wedge \neg(\exists v \,.\, u \rightarrow_R v)$

Example 2

$r_1 : T \wedge x \to x$

$r_2 : F \wedge x \to F$

$r_3 : T \vee x \to T$

$r_4 : F \vee x \to x$

$r_5 : \text{not}(T) \to F$

$r_6 : \text{not}(F) \to T$

$(T \vee F) \wedge \text{not}(F)$

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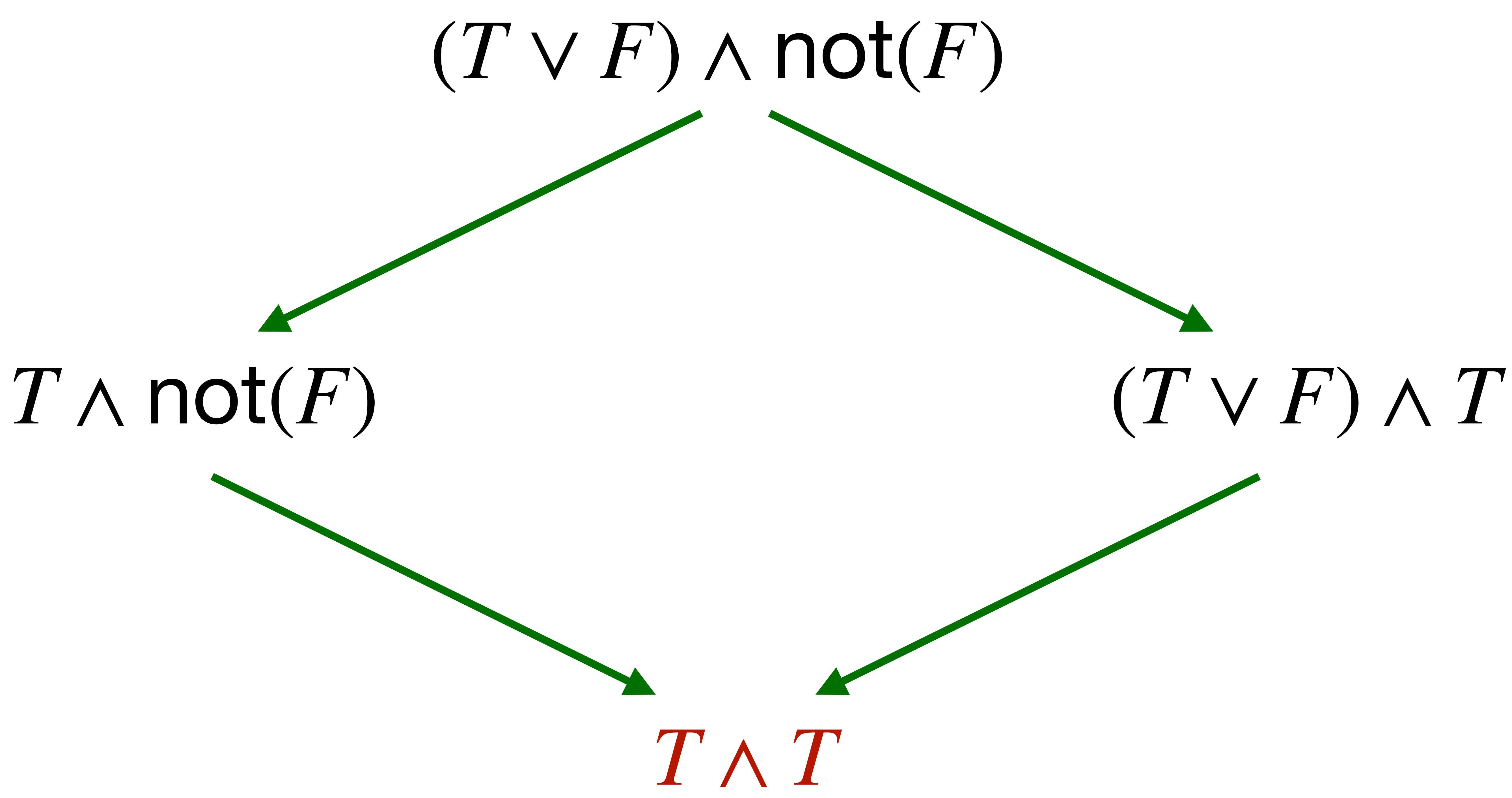

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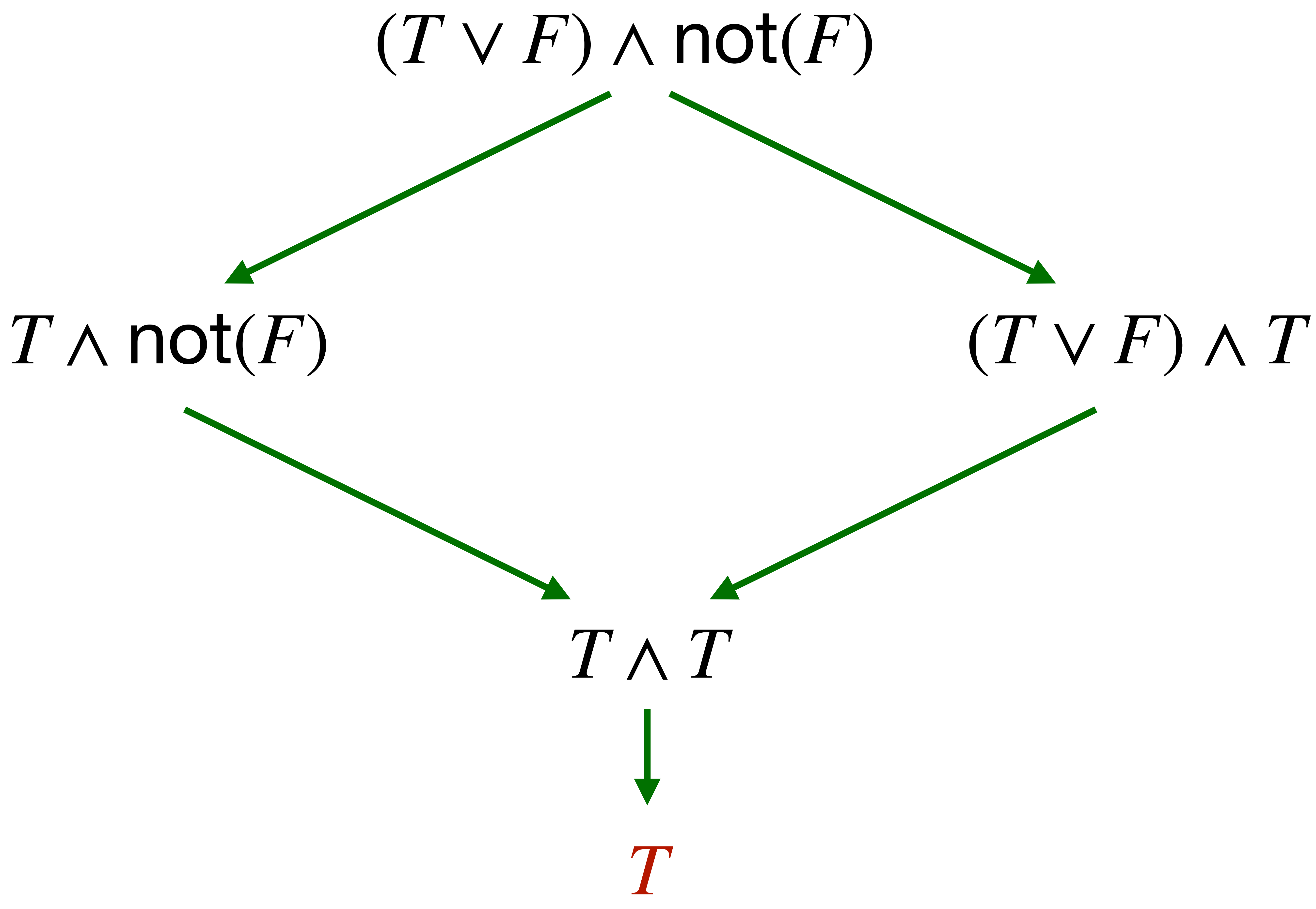

Real-World Applications of Term Rewriting

- Automated Theorem Proving (Equality Saturation)
  - Egg (Willsey et al, 2021)
- Proving Program Termination
  - AProVe (Giesl, Thiemann, Schneider-Kamp, & Falke, 2004)
- Implementing Decision Procedures for Equational Theories
  - Knuth-Bendix Completion (Knuth & Bendix, 1983)

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Confluence

Informally, a TRS is *confluent* if the ordering of the rewrite steps do not matter.

Term rewriting is nondeterministic:

- A single rewrite rule might apply at different locations in a term
- Two rewrite rules may apply to the same term

Formally:

$$t \to_R^* u_1 \land t \to_R^* u_2 \implies \exists w \,.\, u_1 \to_R^* w \land u_2 \to_R^* w$$

Confluence

$$s(z + z) + z$$

Confluence

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$C = s(\square) + z$

$$s(z) + z$$

Confluence

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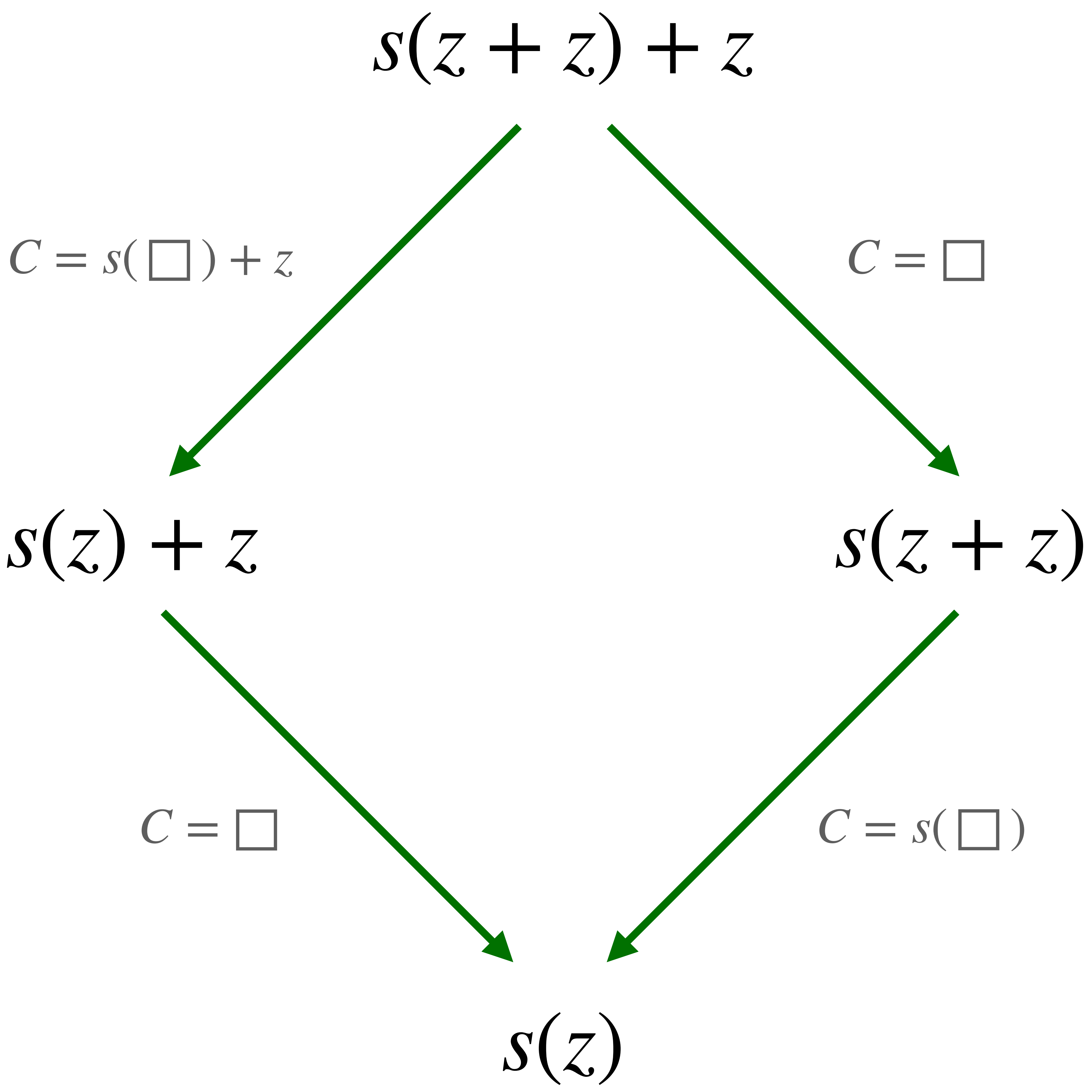

Termination

A TRS (Σ, R) is terminating if there do not exist infinite chains of the form:

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We could then have:

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Why do we care?

- If a TRS for a given theory is terminating and confluent, then it defines a decision procedure for equality in that theory
  - Simply obtain the normal form for each term and compare
  - For more info see (Knuth & Bendix, 1983)
- Various techniques exist for proving termination of rewriting
  - If you want to prove your program terminates, express it as a TRS and prove termination of the TRS

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$$t_1 \to_R t_2 \to_R \cdots$$

Is it possible to decide in general?

No! You can implement a Turing machine as a TRS!

In fact, a Turing machine can be implemented as a single rewrite rule.

Termination

A TRS (Σ, R) is terminating if there do not exist infinite chains of the form:

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Is it possible to decide sometimes?

A Simple Recipe for Proving Termination

1. Define a function $S : T \rightarrow \mathbb{N}$

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Why this works:

For any initial term t, then $S(t)$ is an arbitrary integer n.

There are $n - 1$ numbers that are less than n, so the longest path starting at t has at most n steps.

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$S(t) > S(t) - 2$

$S(t) > S(t) - S(\sigma(x)) - 1$

$S(t) > S(t) - S(\sigma(x)) - 1$

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A General Recipe for Proving Termination

- Recall: The simple recipe worked by mapping elements of T to elements of $\mathbb{N}$
- An infinite descent of $\rightarrow_R$ on T would correspond to an infinite descent of $>$ on $\mathbb{N}$
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- Infinite descent of $>$ on $\mathbb{N}$ is not allowed because $>$ is *well-founded* on $\mathbb{N}$.
- Thus the general recipe for proving termination is:
    - Show that an infinite computation would correspond to an infinite descent in a well-founded relation

Conclusion

- Term Rewriting Systems can serve as expressive yet simple model of computation
- TRS admit interesting properties such as confluence and termination
- In some cases it is possible to prove termination of a TRS

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