GROUP HOMEWORK 3, CPSC 421/501, FALL 2021

JOEL FRIEDMAN

Please note:

(1) You must justify all answers; no credit is given for a correct answer without justification.
(2) Proofs should be written out formally.
(3) Homework that is difficult to read may not be graded.
(4) You may work together on homework in groups of up to four, **but you must submit a single homework as a group submission under Gradescope.**

These exercises use the terms *injection, bijection, surjection*, and, at times, *(one-to-one) correspondence* (a synonym for bijection); these are terms are defined just below Definition 4.2 (page 203) of [Sip]. In addition, these exercises use the $O(\ )$ "big-oh" and $o(\ )$ "little-oh" notation in Section 7.1 (specifically pages 276–278 of [Sip]), although likely you have seen this notation in a previous course on algorithms.

(1) Exercise 8.4.1 on the handout "Uncomputability OR Ruining the Suprises in CPSC421/501."

(2) Exercise 8.4.2 on the handout "Uncomputability OR Ruining the Suprises in CPSC421/501."

(3) Exercise 8.4.3 on the handout "Uncomputability OR Ruining the Suprises in CPSC421/501."

(4) Let $\Sigma = \{a\}$ be an alphabet consisting of the single letter, a. Hence a language over Σ is just subset of

$$\Sigma^* = \{\epsilon, a^1, a^2, a^3, \ldots\}$$

where $\epsilon = a^0$, as usual, denotes the empty string/word.

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(a) Let $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ denote the non-negative integers. **Briefly** explain why the $\mathrm{POWER}(\mathbb{Z}_{\geq 0})$ (i.e., the set of subsets of $\mathbb{Z}_{\geq 0}$) is in one-to-one correspondence with set of languages over Σ, specifically by the correspondence (i.e., bijection)

$$\mathcal{A} \colon \mathrm{POWER}(\mathbb{Z}_{\geq 0}) \to \mathrm{POWER}(\Sigma^*)$$

given by

$$\mathcal{A}(I) = \{a^n \mid n \in I\}.$$

For example,

$$\mathcal{A}\big(\{\text{the positive, even integers}\}\big) = \{a^2, a^4, a^6, \ldots\},$$

and

$$\mathcal{A}\big(\{2, 14\}\big) = \{a^2, a^{14}\}.$$

(b) **Briefly** explain why if $I \subset \mathbb{Z}_{\geq 0}$, then I is finite iff $\mathcal{A}(I)$ is finite, and if so, then $|I| = |\mathcal{A}(I)|$.

(c) We say that $I \subset \mathbb{Z}_{\geq 0}$ is *eventually periodic* if there exists $n_0 \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{N}$ such that for each integer $n \geq n_0$ we have

$$n \in I \iff n + p \in I,$$

i.e., n is in I iff (if and only if) $n + p \in I$. **Briefly** explain why the following subsets of $\mathbb{Z}_{\geq 0}$ are eventually periodic:

(i) any finite subset of $\mathbb{Z}_{\geq 0}$;
(ii) the odd integers greater than 2021;
(iii) the natural numbers divisible by 4, except those divisible by 100 but not by 400 (these are, in a sense, the "leap years" in the Gregorian calendar).

(d) In class (likely on Tuesday, October 5) we will explain why for all $I \subset \mathbb{Z}_{\geq 0}$, I is eventually periodic iff $\mathcal{A}(I)$ is a regular language[1]. You may assume this fact to solve the following exercises.

(i) Show that

$$\{a^n \mid n \text{ is a positive integer that is not a power of } 10\}$$

is not a regular language.

(ii) Fix an $L \subset \Sigma^*$, and for $n \in \mathbb{N}$ let $\pi_L(n)$ denote the number of words of length at most n. Show that if $L \subset \Sigma^*$ is an infinite language for which $\pi_L(n) = o(n)$, then L is non-regular (i.e., not regular).

(iii) Show that if

$$L = \{a^{(n^2)} \mid n \in \mathbb{N}\} = \{a, a^4, a^9, a^{16}, \ldots\}$$

then $\pi_L(n) = O(\sqrt{n}\,)$; conclude that L above is non-regular.

(iv) It is well-known that if

$$L = \{a^n \mid n \text{ is a prime number}\} = \{a^2, a^3, a^5, a^7, a^{11}, \ldots\}$$

then $\pi_L(n) = n/log(n) + o(n/\log(n))$ (this is called the "Prime Number Theorem"). Conclude that L above is non-regular.

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[1] You might think of why this is true: consider what a DFA, $M = (Q, \Sigma, \delta, q_0, F)$, "looks like" with $\Sigma = \{a\}$: roughly speaking, it looks like a *directed graph* where each vertex has *outdegree* equal to 1 (see [Sip], pages 12–13, for this terminology). What does such a DFA look like when you draw it?

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA.
E-mail address: jf@cs.ubc.ca
URL: http://www.cs.ubc.ca/~jf