CPSC 421/501   Nov 19, 2020

- Some more NP-complete problems
- How to solve P vs. NP (?)   §9.3 [Sip]
  "Circuit complexity"
  - There are 2^{2^n} Boolean functions on n-variables
  - For n large, less than $\sqrt{2^{2^n}}$ can be computed by a circuit of size $2^n/2n$
  - If 3COLOUR $\in$ P, then 3COLOUR has polynomial size circuits; can we refute the conclusion???
  - Questions about circuit/formula size/depth of Boolean functions are still wide open.

Breakout room problems:

(1) Show that

$$\text{PARTITION} = \left\{ \langle n_1, \ldots, n_m \rangle \;\middle|\; \text{for some } I \subset [m], \; \sum_{i \in I} n_i = \sum_{i \notin I} n_i \right\}$$

is NP-complete

(2) Show that

$$\text{VERTEX-EXPANSION} = \left\{ \langle G, a, b \rangle \;\middle|\; \text{there is a } A \subset V_G \text{ with } |A| = a \text{ and } |\Gamma(A)| \geq b \right\}$$

is NP-complete

3) Show that CLIQUE $= \{ \langle G, k \rangle \mid G$ has a clique of size $k \}$ is NP-complete

4) Show that every Boolean function on n variables can be computed by a formula of

a) Size $\leq n 2^n$

AND

b) Depth $\leq \lceil \log_2 n \rceil\, n$

Graph theory terminology:

Let $G=(V,E)$ be an (undirected) graph. Let $A \subset V$ be a subset.

(1) $\Gamma(A)$ = "neighbours of A"

$= \{ v \in V \mid v \notin A$ and some edge is incident upon v and some element of $A \}$

(2) A is a clique if any two elements of A are joined by an edge.

Today! - Some more NP-complete problems

- Circuit size for Boolean functions

9.3 [Sip]

Next 2 Weeks

24	26	Nov
1	3	Dec

1 short presentation

10-15 minutes

Rest

presentation, review for final

mostly presentations

Boolean formulas, sums of integers

So far: $\underbrace{\text{SAT, 3SAT}}_{\text{Boolean formulas}}$, $\underbrace{\text{SUBSET-SUM}}_{\text{sums of integers}}$

are NP complete

Today: $\underbrace{\text{CLIQUE}}_{\text{new idea in graph thy}}$ PARTITION

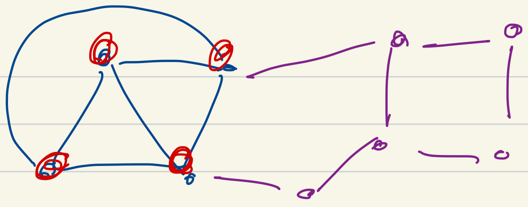
$\frac{1}{4}$CLIQUE $\frac{1}{2}$CLIQUE

MINESWEEPER is also NP-complete

(the proof requires "gadgets" that are pretty sophisticated)

CLIQUE: $\{\langle G, k\rangle \mid G$ has a clique of size $k\}$

clique in $G = (V, E)$ is a subset $A \subset V$ s.t. every two vertices in A is connected by some edge.



clique of size 4

Thm: CLIQUE is NP-complete

Proof: (1) CLIQUE is in NP —

algorithm: given $\langle G, k \rangle$, "non-deterministically guess/write down" a subset of k vertices, and then verify that this subset is a clique.

$G = (V, E)$, $V = [n]$, $k \le n$, and need to check whether or not G contains some $\binom{k}{2}$ edges. $\le \binom{n}{2} \le n^2$.

(2) Show $3SAT \leq_P CLIQUE$

e.g., $(x_1 \text{ OR } x_2 \text{ OR } \neg x_3)$ AND

$(x_1 \text{ OR } \neg x_2 \text{ OR } x_4)$ AND

$(\neg x_2 \text{ OR } x_3 \text{ OR } \neg x_4)$

$\rightsquigarrow$ produce a graph that finding a clique of a certain $\iff$ is satisfiable

Hint to construction:

x_1 $\quad$ x_2 $\quad$ $\neg x_3$ $\quad \leftarrow$ 1st clause

x_1 $\qquad\qquad\qquad$ $\neg x_2$

$\neg x_2$ $\qquad\qquad\qquad$ x_3

x_4 $\qquad\qquad\qquad$ $\neg x_4$

Now put in edges s.t. there's a 3 clique iff

Another type of reduction:

PARTITION $= \{ \langle n_1, \dots, n_m \rangle \mid$ s.t.

there is $I \subset \{1, \dots, m\}$ s.t.

$$\sum_{i \in I} n_i = \sum_{j \notin I} n_j \}$$

e.g. $\langle 3, 4, 5, 12 \rangle \in$ PARTITION

$\langle 1, 2, 3, 1000 \rangle \notin$

To show PARTITION is NP-complete

① Guess $I \subset \{1, \dots, m\}$, then check if ↗

② SUBSET-SUM $\leq_p$ PARTITION

given SUBSET-SUM question: $\langle n_1, \dots, n_k ; t \rangle$

now create a PARTITION instance

equivalent to $\langle n_1, \ldots, n_k ; t \rangle$ in SUBSET-SUM

is there $K \subset [k] = \{1, \ldots, k\}$

s.t.

$$\sum_{i \in K} n_i = t$$

given

$\rightsquigarrow$ partition problem

"Guess..." = "non-deterministically write down..."

Breakout!

Problems (1) and/or (3)

10:10 $\rightarrow$ 10:20

CLIQUE: e.g., $(x_1 \text{ OR } x_2 \text{ OR } \neg x_3)$ AND
$(x_1 \text{ OR } \neg x_2 \text{ OR } x_4)$ AND
$(\neg x_2 \text{ OR } x_3 \text{ OR } \neg x_4)$

T T T T

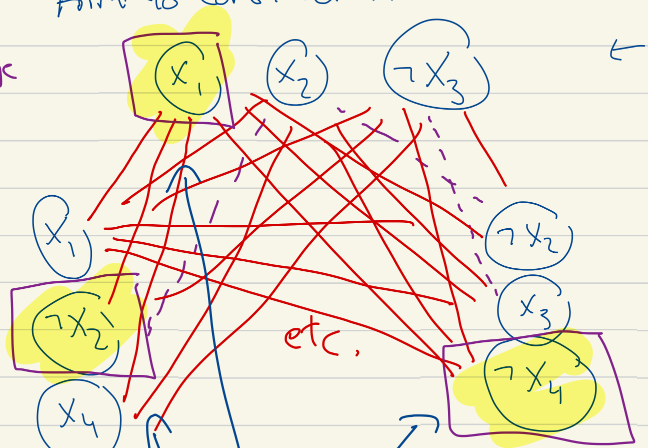
Hint to construction:

① ------ no edge if opposite

② no edge if in same clause

③ create from this, put in other edges

← 1st clause



Now put in edges s.t. there's a 3 clique iff

form a clique size 3

if 3CNF is satisfiable, then graph has a clique

" " " not " , " " doesn't have a clique

if graph has a clique $\Rightarrow$ 3CNF is satisfiable

SUBSET-SUM : $\langle 1, 3, 7, 19, 4, 31; 57 \rangle$

$\langle n_1, n_2, n_3 \ldots, n_k ; t \rangle$

$\rightarrow$ PARTITION

1 3 7 19 4 31 $\rightarrow$ 57

(also $1+3+7+19+4+31=65$)

$\underbrace{1 \quad 3 \quad 7 \quad 19 \quad 4 \quad 31}_{\text{can you split}}$ 57

t

t 57 and 8 sum - t

2t-S

1 3 7 19 4 31 57

add 1008 — 1000 — ??

add

2t-S

1 3 7 19 4 31 32 33 34 35 ; 57

$$(sum - t) + 3(sum + t) + \underbrace{3(sum + t)}_{\text{suff large}}$$

81

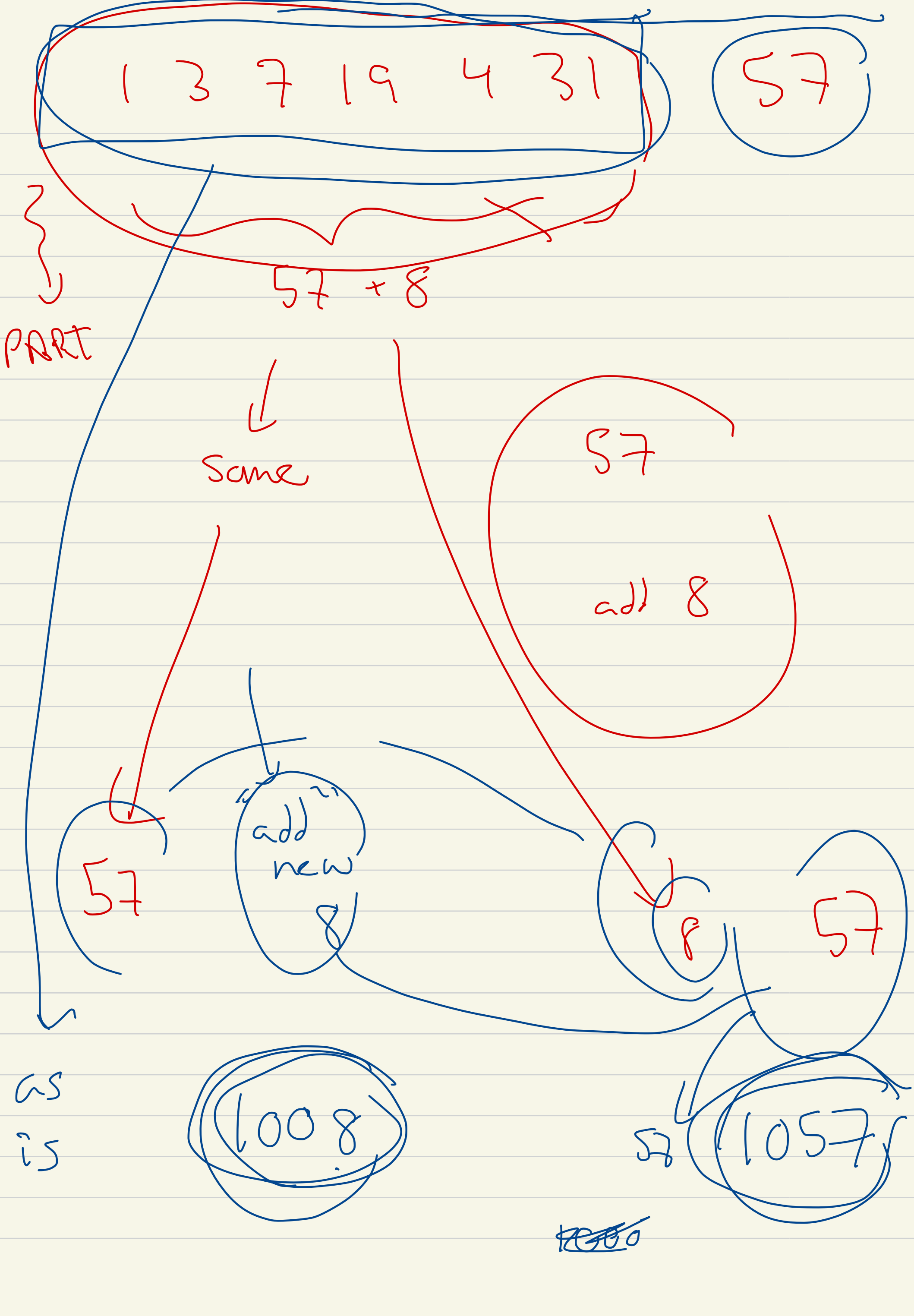
$$\underbrace{10 \quad 30 \quad 70 \;|\; 90 \quad 40 \quad 310}_{57 + 8} \qquad 571$$

t

81 $\qquad$ 571

8 { some numbers | rest } 57

sum is the same


1 3 7 19 4 31
57
PART
57 + 8
same
57
add 8
57
add new 8
8
57
as is
1008
57
1057

Class ends