CPSC 421/501 Nov. 12, 2020

- Finish the Cook-Levin theorem:
  - if SAT $\in$ P, then NP = P
  - if 3SAT $\in$ P, then NP = P

- In practice we are more interested in 3SAT

[We will need some Boolean algebra...]

- Formalize NP-completeness and reductions

- Many languages are NP-complete:

  3SAT, 3-COLOUR, 4-COLOUR, etc.

  SUBSET-SUM, PARTITION, etc.

  EXPANSION, etc.

  VERTEX COVER, etc.

  etc.

Breakout Room Problems

(1) Say that SAT $\in$ P. Show that given a Boolean formula, $f = f(x_1, \ldots, x_n)$, one can find $x_1^*, \ldots, x_n^* \in \{T, F\}$ s.t. if $f \in$ SAT, then $f(x_1^*, \ldots, x_n^*) = T$.

(2) If $L_1 \leq_p L_2$ by an $O(n^3)$ reduction,

and $L_2 \leq_p L_3$ " " $O(n^5)$ " ,

then $L_1 \leq_p L_3$. How much time does the reduction require?

(3) Say that 3COLOUR is NP-complete.

Show that 4COLOUR " " " .

Is 2COLOUR NP-complete?

(4) Show that for fixed $x_1, \ldots, x_6$,

$$x_1 \text{ OR } x_2 \text{ OR } x_3 \text{ OR } x_4 \text{ OR } x_5 \text{ OR } x_6 = T$$

iff (for those values of $x_1, \ldots, x_6$) the formula

$$(x_1 \text{ OR } x_2 \text{ OR } y_1) \text{ AND}$$
$$(\neg y_1 \text{ OR } x_3 \text{ OR } y_2) \text{ AND}$$
$$(\neg y_2 \text{ OR } x_4 \text{ OR } y_3) \text{ AND}$$
$$(\neg y_3 \text{ OR } x_5 \text{ OR } x_6)$$

is satisfiable.

(5) Say that $L \in NP$ and we can prove that $L \in P \Rightarrow NP = P$. Does this mean L is necessarily NP-complete?

(6) Any Boolean $f = f(x_1, \dots, x_n)$ can be written as

$$(\text{clause}_1) \text{ OR } (\text{clause}_2) \text{ OR } \dots \text{ OR } (\text{clause}_{2^n})$$

where

$$\text{clause}_i = \text{literal}_{i,1} \text{ AND } \text{literal}_{i,2} \text{ AND } \dots \text{ AND } \text{literal}_{i,n}$$

where each $\text{literal}_{i,j}$ is one of

$x_1, x_2, \dots, x_n, \neg x_1, \neg x_2, \dots, \neg x_n$

(and $\neg$ is negation),

i.e. as a DNF of size 2^n (or less) and width n (or less).

(7) Any Boolean $f = f(x_1, \dots, x_n)$ can be written in CNF of size 2^n and width n.

"The only constant that a theoretician care about is the one in their salary"

— Richard Karp

[We'll see this today...]

Idea: We have a non-deterministic Turing machine / algorithm M, we have an input to M, w, given $\langle M, w \rangle$

(assuming $Q = \{1, \ldots, |Q|\}$, $\Gamma = \{1, \ldots, |\Gamma|\}$, in Q

1 = initial state
2 = accept state
3 = reject "

Convert $\langle M, w \rangle \xrightarrow{\text{"reduction"}}$ Boolean formula $f_{M,w}$

s.t.

1. M accepts w iff $f_{M,w} \in SAT$

2. $f_{M,w}$ has size poly in $|w|$

3. We can produce $f_{M,w}$ with a poly time algorithm

there are huge constants

$O(\text{poly}(n))$

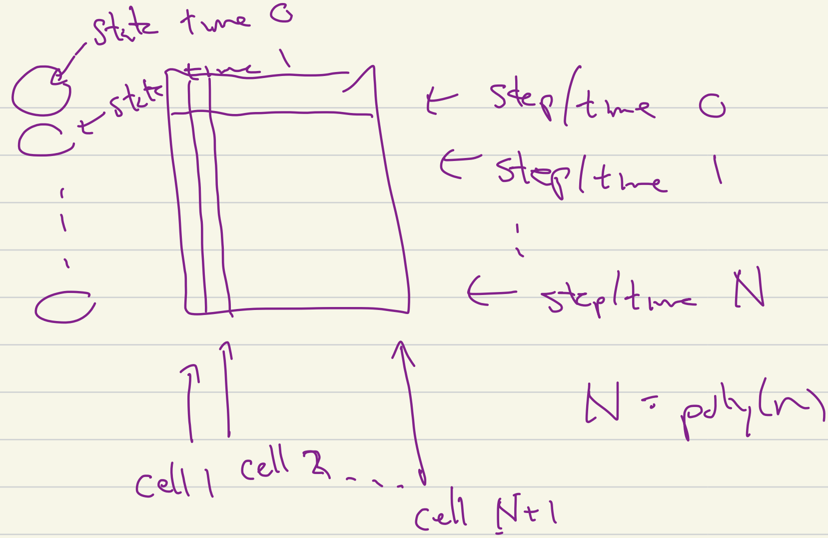
uh-oh

(U. Manber)

4. Then, if $SAT \in P \Rightarrow$ Language$(M) \in P$

This proof really gives a "reduction"

Idea:



$$X_{ijk} = \begin{cases} T & \text{if at time } i\text{, cell } j\text{, the symbol } k \text{ is written} \\ F & \text{otherwise} \end{cases}$$

$$Y_{ij} = \begin{cases} T & \text{if at time } i\text{, tape head is at cell } j \\ F & \text{otherwise} \end{cases}$$

$$Z_{is} = \begin{cases} T & \text{at time } i\text{, we are in state } s \\ F & \text{otherwise} \end{cases}$$

We want for all i, j, exact one T among

(1) $X_{i,j,1}, X_{i,j,2}, \ldots, X_{i,j,|\Gamma|}$

AND

(2) similarly for Y_{ij}, Z_{is}

AND

(3) We want $Z_{N,2} = T$

(i.e. after N steps we are in the accept state)

AND

(4) We want $w_1 \cdots w_n \sqcup \sqcup \sqcup$ to be written initially, and initially tape head in cell 1, state = initial state

AND

(5) The transition from step i to step i+1 is a valid computation in M (for $i = 0, 1, 2, \dots, N-1$)

Remark: Ultimately, we want have formula $f_{M,w}$ to be in 3CNF form! i.e.

$f_{m,w}$ = clause_1 AND clause_2 AND ---

AND clause_{M} , (M poly in $|w|$)

where

clause_i = $\text{literal}_{i,1}$ OR $\text{literal}_{i,2}$ OR $\text{literal}_{i,3}$

where literal = $x_1, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n$

AND = $\wedge$ OR = $\vee$, NEGATION = $\neg$ (NOT)

---

3) $z_{N,2} = 1$ is same as

$(z_{N,2} \text{ OR } z_{N,2} \text{ OR } z_{N,2})$

② Part! for all i,

$z_{i,1}, z_{i,2}, z_{i,3}, \ldots, z_{i,|Q|}$

exactly one is true.

$= (z_{i,1} \text{ OR } z_{i,2} \text{ OR } z_{i,3} \text{ OR} \ldots \text{ OR } z_{i,|Q|})$

(is true) need trick → 3 CNF

AND

$= (\neg z_{i,1} \text{ OR } \neg z_{i,2} \text{ OR } \neg z_{i,2})$

AND

$(\neg z_{i,1} \text{ OR } \neg z_{i,3} \text{ OR } \neg z_{i,3})$

⋮

i.e. $\text{AND}_{j_1 < j_2} \underbrace{(\neg z_{i,j_1} \text{ OR } \neg z_{i,j_2} \text{ OR } \neg z_{i,j_2})}^{\text{3 literals}}$

Trick: Think of $z_{i,1}, \ldots, z_{i,|Q|}$ as fixed; then

$$z_{i,1} \text{ or } z_{i,2} \text{ or } \ldots\ z_{i,|Q|} = \overline{\top}$$

iff this is satisfiable (add $u_{i,\ell}$):

$(z_{i,1} \text{ or } z_{i,2} \text{ or } u_{i,1})$

AND

$(\neg u_{i,1} \text{ or } z_{i,2} \text{ or } u_{i,2})$

auxilliary dummy variables

AND

$(\neg u_{i,2} \text{ or } z_{i,3} \text{ or } u_{i,3})$

$\vdots$

$(\neg u_{i,|Q|-1} \text{ or } z_{i,|Q|-1} \text{ or } z_{i,|Q|})$

Breakout rooms for problems

4 & 6

10:23 – 10:31

6 is DNF, not CNF

(4)

X_1 or X_2 or X_3 or X_4 or X_5 or X_6 = T

iff (for those values of $X_1, \ldots, X_6$) the formula

$$\boxed{\begin{aligned}&(X_1 \text{ or } X_2 \text{ or } Y_1) \text{ AND}\\&(\neg Y_1 \text{ or } X_3 \text{ or } Y_2) \text{ AND}\\&(\neg Y_2 \text{ or } X_4 \text{ or } Y_3) \text{ AND}\\&(\neg Y_3 \text{ or } X_5 \text{ or } X_6)\end{aligned}}$$

is satisfiable.

Say

$$\left.\boxed{\begin{aligned}&(X_1 \text{ or } X_2 \text{ or } Y_1) \text{ AND}\\&(\neg Y_1 \text{ or } X_3 \text{ or } Y_2) \text{ AND}\\&(\neg Y_2 \text{ or } X_4 \text{ or } Y_3) \text{ AND}\\&(\neg Y_3 \text{ or } X_5 \text{ or } X_6)\end{aligned}}\right\}$$

can be satisfied

but $X_1 = f, X_2 = f, \ldots, X_n = f$

then

$$\left.\begin{aligned}&X_1 = F,\ X_2 = f,\ \cancel{\text{and}} \Rightarrow Y_1 = T\\&\neg Y_1 = F, \text{ and } X_2 = f \Rightarrow Y_2 = T\\&\neg Y_2 = f,\ \ldots\ X_3 = f \Rightarrow \ldots\end{aligned}\right\}\ \text{contradiction}$$

So \ blah can be satisfied, at least one of $x_1, \ldots, x_n$ must be T

But if, say, $x_3 = T$ then

true $y_1 = T$

$(x_1 \text{ or } x_2 \text{ or } y_1)$ AND

$\neg y_1 = F$ $(\neg y_1 \text{ or } x_3 \text{ or } y_2)$ AND true

$(\neg y_2 \text{ or } x_4 \text{ or } y_3)$ AND $y_2 = f$

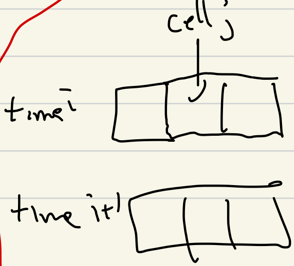
true

$(\neg y_3 \text{ or } x_5 \text{ or } x_6)$ $y_3 = f$

true ⋮

~~Redraw~~

(5) step $i \rightarrow$ step $i+1$ is valid transition in M

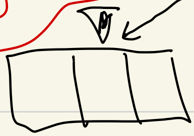


if tape head isn't at cell j, time i

then contents cell j, time $i+1$

= " " " " :

AND

time i  cell j

if tape head at time i is on cell j,

then

if state 1

then exclusively one of

cell j, time i+1: write poss$_1$

next state poss_state$_1$

tape head position poss_tape$_1$

OR

write poss$_2$

next state poss_state$_2$

tape head poss_tape$_2$

OR

⋮

if state 2

( )

⋮

this can be written as some 3 CNF (huge constant)