Finish the Cook-Levin theorem:
- if SAT ∈ P, then NP = P
- if 3SAT ∈ P, then NP = P

In practice we are more interested in 3SAT

[We will need some Boolean algebra...]

Formalize NP-completeness and reductions

Many languages are NP-complete:
- 3SAT, 3-COLOUR, 4-COLOUR, etc.
- SUBSET-SUM, PARTITION, etc.
- EXPANSION, etc.
- VERTEX COVER, etc.
- etc.
Breakout Room Problems

1. Say that \textsc{SAT} is \textsc{P}. Show that given a Boolean formula, \( f = f(x_1, \ldots, x_n) \), one can find \( x_1^*, \ldots, x_n^* \in \{ T, F \} \) s.t. if \( f \in \textsc{SAT} \), then \( f(x_1^*, \ldots, x_n^*) = T \).

2. If \( L_1 \leq_p L_2 \) by an \( O(n^3) \) reduction, and \( L_2 \leq_p L_3 \) \( \Rightarrow \) \( O(n^5) \), then \( L_1 \leq_p L_3 \). How much time does the reduction require?

3. Say that \textsc{3COLOUR} is \textsc{NP}-complete. Show that \textsc{4COLOUR} \( \leq_p \) \textsc{3COLOUR}. Is \textsc{2COLOUR} \textsc{NP}-complete?
4) Show that for fixed $x_1, \ldots, x_6$, $x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5 \lor x_6 = T$ iff (for those values of $x_1, \ldots, x_6$) the formula

$$(x_1 \lor x_2 \lor y_1) \land (\neg y_1 \lor x_3 \lor y_2) \land (\neg y_2 \lor x_4 \lor y_3) \land (\neg y_3 \lor x_5 \lor x_6)$$

is satisfiable.

5) Say that $L \leq NP$ and we can prove that $L \leq P \Rightarrow NP = P$. Does this mean $L$ is necessarily $NP$-complete?
Any Boolean \( f = f(x_1, \ldots, x_n) \) can be written as

\( \text{(clause}_1\text{)} \) OR \( \text{(clause}_2\text{)} \) OR \( \ldots \) OR \( \text{(clause}_2^n\text{)} \)

where

\( \text{clause}_i = \text{literal}_{i,1} \text{ AND literal}_{i,2} \text{ AND} \ldots \text{ AND literal}_{i,n} \)

where each literal \( i,j \) is one of

\( x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n \)

(and \( \neg \) is negation),

i.e., as a DNF of size \( 2^n \) (or less) and width \( n \) (or less).

Any Boolean \( f = f(x_1, \ldots, x_n) \) can be written in CNF of size \( 2^n \) and width \( n \).
"The only constant that a theoretician care about is the one in their salary"

- Richard Karp

[We'll see this today...]

Idea: We have a non-deterministic Turing machine / algorithm $M$, we have an input to $M$, $w$, given $\langle M, w \rangle$

(assuming $Q = \{1, \ldots, |Q|\}$, $
\Gamma = \{1, \ldots, |\Gamma|\}$, in $\Omega$

$1 = \text{initial state}$
$2 = \text{accept state}$
$3 = \text{reject} \)
Convert $\langle M, w \rangle \rightarrow$ Boolean formula $f_{M,w}$

s.t. (1) $M$ accepts $w$ iff $f_{M,w} \in \text{SAT}$

there are huge constants $0(\text{poly}(n))$ a poly time algorithm

which (4) then, if SAT $\in P \Rightarrow$

Language$(M) \in P$

This proof really gives a "reduction"
Idea: Consider a tape and a Turing machine. Let $T$ be the tape at time $i$, $j$ be the cell at time $i$, and $k$ be the symbol written.

Let $X_{ijk} = \begin{cases} T & \text{if at time } i, \text{ cell } j, \text{ the symbol } k \text{ is written} \\ F & \text{otherwise} \end{cases}$

Let $Y_{ij} = \begin{cases} T & \text{if at time } i, \text{ tape head is at cell } j \\ F & \text{otherwise} \end{cases}$

Let $Z_{is} = \begin{cases} T & \text{if time } i, \text{ we are in state } s \\ F & \text{otherwise} \end{cases}$

We want for all $i, j$, exactly one $T$ among $X_{ij1}, X_{ij2}, \ldots, X_{ijm}$.
Similarly for $Y_{ij}$, $Z_{ij}$

We want $Z_{N/2} = T$

(i.e. after $N$ steps we are in the accept state)

We want $w_1 \ldots w_n$ to be written initially, and initially tape head in cell 1, state = start state

The transition from step $i$ to step $i+1$ is a valid computation in $M$ (for $i = 0, 1, 2, \ldots, N-1$)

Remark: Ultimately, we want have formula $f_{m,w}$ to be in 3CNF form! i.e.
\[ f_{m,w} = \text{clause}_1 \land \text{clause}_2 \land \ldots \land \text{clause}_M, \quad (M \text{ poly.}) \]

where

\[ \text{clause}_i = \text{literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_Q \]

where

\[ \text{literal}_j = x_1, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n \]

AND = \land \quad \lor = \lor, \quad \text{NEGATION} = \neg

(\text{NOT})

(3) \[ Z_{N,2} \Rightarrow \bar{l} \text{ is same as} \]

\[ (Z_{N,2} \lor Z_{N,2} \lor Z_{N,2}) \]
(2) Part: for all \( i \),

\[ z_{i,1}, z_{i,2}, z_{i,3}, \ldots, z_{i,10} \]

exactly one is true.

\[ (\text{is true}) \quad \text{need trick} \rightarrow \text{3 CNF} \]

\[ \neg \neg z_{i,1} \text{ or } \neg \neg z_{i,2} \text{ or } \neg \neg z_{i,3} \ldots \]

\[ \text{AND} \]

\[ \neg \neg z_{i,1} \text{ or } \neg \neg z_{i,2} \text{ or } \neg \neg z_{i,3} \ldots \]

\[ \text{AND} \]

\[ \neg z_{i,1} \text{ or } \neg z_{i,2} \text{ or } \neg z_{i,3} \ldots \]

\[ \text{3 literals} \]

\[ \text{i.e. AND} \]

\[ \neg z_{i,j} \text{ or } \neg z_{i,j} \text{ on } j < j \]

\[ \neg z_{i,j} \text{ or } \neg z_{i,j} \text{ on } j \]
Trick: Think of $Z_{i,1}, \ldots, Z_{i,q_l}$ as fixed; then

$$Z_{i,1} \lor Z_{i,2} \lor \neg Z_{i,1} = 1$$

iff this is satisfiable (add $U_{i,x}$):

$$(Z_{i,1} \lor Z_{i,2} \lor U_{i,1})$$

AND

$$(\neg U_{i,1} \lor Z_{i,2} \lor U_{i,2}) \text{ variables}$$

AND

$$(\neg U_{i,2} \lor Z_{i,3} \lor U_{i,3})$$

AND

$$(\neg U_{i,3} \lor Z_{i,4} \lor Z_{i,1_{q-1}} \lor Z_{i,1_{q+1}})$$
Breakout rooms for problems

④ & ⑥

10:23 - 10:31

⑥ is DNF, not CNF
$X_1 \lor X_2 \lor X_3 \lor X_4 \lor X_5 \lor X_6 = T$

iff (for those values of $X_1$ to $X_6$) the formula

$(X_1 \lor X_2 \lor Y_1)$ AND
$(\neg Y_1 \lor X_3 \lor Y_2)$ AND
$(\neg Y_2 \lor X_4 \lor Y_3)$ AND
$(\neg Y_3 \lor X_5 \lor X_6)$

is satisfiable.

Say

$(X_1 \lor X_2 \lor Y_1)$ AND
$(\neg Y_1 \lor X_3 \lor Y_2)$ AND
$(\neg Y_2 \lor X_4 \lor Y_3)$ AND
$(\neg Y_3 \lor X_5 \lor X_6)$

but $X_1 = f, X_2 = f, \ldots, X_6 = f$

can be satisfied

then $X_1 = f, X_2 = f, \ldots$ $Y_1 = T$

$\neg Y_1 = f, \ldots$ $Y_2 = T$

contradict
So I blab can be satisfied, at least one of $x_1, \ldots, x_n$ must be $T$

But if, say, $x_3 = T$ then

\[
\begin{align*}
\neg y_1 &= F \\
\neg y_2 &= F \\
\neg y_3 &= F
\end{align*}
\]

true $y_i = T$

\[
\begin{align*}
(X_1 \lor x_2 \Rightarrow y_1) \land
\neg y_1 &= 0 \\
\neg y_1 \land x_3 \Rightarrow y_2) \land
\neg y_2 &= 0 \\
\neg y_2 \land x_4 \Rightarrow y_3 \land
\neg y_3 &= 0
\end{align*}
\]

\(\vdots\)

\(YET\)

\(\therefore\)

\(\therefore\)

Below

(5) step $i = step_{i+1}$ is valid transition in $M$

\[
\text{time } i
\]

\[
\text{time } i+1
\]

if tape head isn't at cell $j$, time $i$

then contains cell $j$, time $i+1$
if tape head at time \( i \) is on cell \( j \), then

if state 1

then exclusively one of

- Cell \( j \), time \( i+1 \): write possible
- Next state poss_state,
- Tape head position poss_type,
- OR
- Write poss_z
- Next state poss_state_z
- Tape head poss_tape_z
- OR

if state 2

This can be written as some 3 CNF

(huge const)