CPSC 421/501 Sept 22, 2020

- Set Theory Subtleties

  Not subtle: If T is uncountable, and there is a surjection $S \to T$, then S is uncountable.

  Subtle: If there is an injection $S \to T$, then there is a surjection $T \to S$.

- Russell's Paradox

- Related Paradoxes (Section 6 of handout)

- Start Finite Automata (§1.1 of Textbook)??

BREAKOUT ROOM PROBLEMS

① If S is countable, and there is a surjection $S \to T$, then T is countable. (Prove this)

related

② Is $\text{POWER}(\{a,b\}^*) \cup \mathbb{N}$ countable?

③ Is there a bijection

$$[2]^{\mathbb{N}} \to [3]^{\mathbb{N}},$$

← new notation

and can you describe one?

(4) Let F be the set of ~~functions~~ functions $\mathbb{N} \to \mathbb{N}$ that can be desribed in English (assuming a fixed, precise interpretation of English). Is F countable or uncountable?

related

(5) Let $\mathbb{N}^{\mathbb{N}}$ be the set of functions $\mathbb{N} \to \mathbb{N}$. Is $\mathbb{N}^{\mathbb{N}}$ countable or uncountable?

Today: - Finish the handout

- Breakout rooms (students may be able to form their own breakout rooms, as of yesterday)

---

§5 of handout: Russell's Paradox & Set Theory Subtleties

OR (1) Why we consult with set theorists & logicians

(2) How to find more countable and uncountable sets

Last time:

Cantor's Thm: Any map $S \xrightarrow{f} \text{POWER}(S)$ is not surjection. Idea

$$T = \{ s \in S \mid s \notin f(s) \}$$

kind of a selfref.

negation

is not in the image of f. [Main tool for uncountable sets.]

Implies: $\text{POWER}(\mathbb{N})$ is uncountable

application $\text{POWER}(\Sigma^*)$ " " Σ = alphabet

Textbook, §4.2 proves that $\mathbb{R}$ is uncountable, whereas

$\mathbb{N}$, $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers) are countable

Last time $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2 = \{(a,b) \mid a,b \in \mathbb{N}\}$

is countable

Proof 1:

$$\begin{array}{ccccc} & & (1,1) & & \\ & (2,1) & & (1,2) & \\ (3,1) & & (2,2) & & (1,3) \\ & & \vdots & & \end{array}
\qquad\Bigg|\qquad
\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & & \\ (3,1) & & & \\ \vdots & & & \end{array}$$

Idea

$$\left.\begin{array}{l} (3,5) \rightsquigarrow 35 \\ (17,1) \rightsquigarrow 171 \\ \mathbb{N}^2 \longrightarrow \mathbb{N} \end{array}\right\}$$

Fix: (Gödel numbering)

$$(3,5) \to 2^3 3^5$$
$$(17,1) \to 2^{17} 3^1$$
$$\vdots$$
$$(a,b) \longrightarrow 2^a 3^b$$

Gives map $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ injection

Since $2^a 3^b = 2^{a'} 3^{b'} \Rightarrow \begin{array}{l} a = a' \\ b = b' \end{array}$

Our definition: A is uncountable, if there exist <u>no</u> map $f: \mathbb{N} \to A$ that is surjective.

Here: $\mathbb{N}^2 \to \mathbb{N}$ is <u>injective</u>

$$\boxed{\begin{array}{l}\text{Can you prove: if there is an injection } S \to \mathbb{N}, \text{ is there} \\ \text{a surjection } \mathbb{N} \to S \text{ ?????}\end{array}}$$

This is true for finite sets

$$[3] = \{1,2,3\}, \quad [4] = \{1,2,3,4\}$$

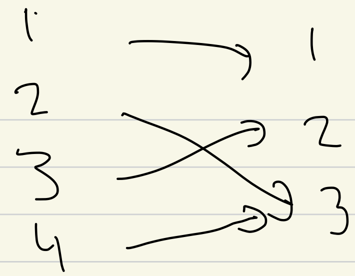
injection $[3] \to [4]$

$1 \to 1$
$2 \to 2$
$3 \to 3$
4

OR $[3] \to [4]$

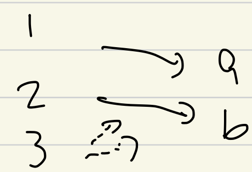
$1 \to 2$
$2 \to 3$
$3 \to 4$
1

and surjection $[4] \to [3]$



<u>Intuition!</u> If there is an injection $S \to T$

"$\text{size}(S) \leq \text{size}(T)$"



<u>Intuition!</u> If there is a surjection $T \to S$

then "$\text{size}(T) \geq \text{size}(S)$"

Is it true that ① $\exists$ surjection $T \to S$?

$\Leftrightarrow$ ② $\exists$ injection $S \to T$

For S, T finite, or $= \mathbb{N}$ yes

" $S, T \rightsquigarrow$ subtle (assumptions on "size" "cardinality")

If we assume this then

$\underbrace{\exists \text{ surjection } \mathbb{N} \to S}_{\text{"countable"}} \Leftrightarrow \underbrace{\exists \text{ injection } S \to \mathbb{N}}_{\text{an equivalent form}}$

Example: $\mathbb{N}^2 \to \mathbb{N}$

$$(a,b) \longmapsto 2^a 3^b \quad \text{is an injection}$$

$\Rightarrow$ $\exists$ a surjection $\mathbb{N} \to \mathbb{N}^2$

To build: look at

$$\mathbb{N}: \quad \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \underset{\times\, 2^a3^b}{\cancel{5}}, 6, \underset{\times\, 2^a3^b}{\cancel{7}}, \cancel{8}, \cancel{9}, \cancel{10}, \cancel{11}, \underset{= 2^2 3^1}{12}, \ldots$$

$$1 \downarrow \quad 2^0 3^0 = 2^{\text{some natural}} \, 3^{\text{some natural}}$$

$$6 \to 2^1 3^1$$

in image of $\mathbb{N}^2 \to \mathbb{N}$

not easy to describe map $\mathbb{N} \to \mathbb{N}^2$ based on the map $\mathbb{N}^2 \to \mathbb{N}$

If S is countable, and there is a surjection $S \to T$, then T is countable

If S is countable, and there is a ~~surjection~~ an injection $\cancel{S \to T}$, then T is countable

$T \to S$

Claim: If T is uncountable and

there is a surjection $S \to T \Rightarrow S$ is uncountable
" " an injection $T \to S \Rightarrow$ " " "

are equivalent

Russell's Paradox: Let R be the set of sets that do not contain themselves.

Does $R \in R$?

Idea of Paradox: can have ① self-reference ② negation

$$R = \{ S \text{ is a set s.t. } S \notin S \}?$$

Either $R \in R$ or $R \notin R$.

① $R \in R$ then R is a set s.t. $R \notin R$. Contradiction

② $R \notin R$ then R is not a set s.t. $R \notin R$, i.e. $R \in R$

also a contradiction

Related paradox: Does the set of all sets contain itself.

Next: we describe a few more $\begin{cases} \text{"paradoxes"} \leftarrow \text{Russell's} \\ \text{"thms"} \leftarrow \text{Cantor's thm.} \end{cases}$

$$[n] = \{1, 2, \ldots, n\}$$

Notation: $\{\text{functions from } S \to T\}$

set of functions

Notation T^S

E.g. $\text{PRIMES} = \{2, 3, 5, 7, 11, \ldots\} \subset \{0, \ldots, 9\}^*$

left out 02, 03

i.e. PRIMES is a language over $\{0, \ldots, 9\} = \Sigma$

alphabet

PRIMES

$0 \longrightarrow$ no
$1 \longrightarrow$ no
$2 \longrightarrow$ yes
$3 \longrightarrow$ yes
$\vdots$
$9 \longrightarrow$ no
00
01

$02 \longrightarrow \begin{cases} \text{no} \\ \text{yes} \end{cases}$

depends on if you allow leading zeros

More generally

equiv

(1) language over Σ

(2) elt of $\mathrm{Power}(\Sigma^*)$

(3) a decision $\Sigma^* \to \{\text{yes}, \text{no}\}$

(4) an element of

$$\{\text{yes}, \text{no}\}^{\Sigma^*} = \{\text{functions } \Sigma^* \to \{\text{yes}, \text{no}\}\}$$

~~10:26 ⟵ 10:43~~

My watch 10:28 am ⟶ 10:43

15 minute breakout, or so

Breakout: (4) & (5)

function $\mathbb{N} \to \mathbb{N}$ describe in English.

"The function taking 1 to 17, 2 to 23, ..."

allow infinite

But finite strings

"The function taking x to sqrt(x)."

↳ interpret

Strings in ASCII $=$ ASCII* countable

⑤ $\{$functions $\mathbb{N} \to \mathbb{N}\}$ uncountable

functions $\mathbb{N} \to \{1, 2\}$

$\Leftrightarrow$ $\mathbb{N} \to \{\text{yes}, \text{no}\}$ bijection

$\Leftrightarrow$ POWER($\mathbb{N}$) uncountable

---

① S countable, surj $S \to T$ then

② T is countable

$\Rightarrow$ surj $S \to T$ and T uncountable then $\mathcal{P}$ uncountable

Next time: finish this discussion

- Resolve Russell's paradox
- 1-2 paradox
- Textbook Ch. 1, Regular Lang.

$\Sigma = \{a, b, c\}$

ε $\qquad \Sigma^0$

$a \quad b \quad c$ $\qquad \Sigma^1$

$aa \quad ab \quad ac \quad \ldots \quad cc$ $\qquad \Sigma^2$

$\mathbb{N} = \{ 1, 2, 3, 4, 5, \ldots \}$

$\{0, 1\}^* = \{ \varepsilon, 0, 1, 00, 01, \ldots \}$

Some function or meaningless

Some function or meaningless