CPSC 421/501 Sept 17, 2020

Topics for today or soon:

Section 4 of handout:

- Countable Sets, Uncountable Sets

- Cantor's Thm

Sections 5 and 6

- Russell's Paradox and Set Theory Subtleties

- Related Paradoxes and Theorems

BREAKOUT ROOM PROBLEMS

(1) Show that $\mathbb{N} \times \mathbb{N}$ is countable

(2) Show that $\mathbb{Z} \times \mathbb{Z}$ is countable

(3) Let $f : [4] \to \text{Power}([4])$

(where $[4] = \{1,2,3,4\}$) be given by:

$f(1) = \emptyset$, $f(2) = \{1,2\}$, $f(3) = \{2,4\}$, $f(4) = [4]$,

Describe $T = \{ s \in [4] \mid s \notin f(s) \}$.

Convince yourself that $T$ is not in the image of $f$. [This is not a precise task.]

④ Is $\text{Power}(\{a,b\}^*) \cup \mathbb{N}$ countable?

⑤ Find an example of an injective and/or surjective and/or bijective map to help you remember these term.

[This is not a precise question.]

e.g. $\{\text{students}\} \xrightarrow[\text{inj}]{} \{\text{ID numbers}\}$

Admin Stuff:

(1) Individual Homework 1 { write up individually, say whom you worked with

Group Homework 1 { Submit as a group, same grade as your group members

You can work in groups of up to 4

Gradescope ~~sign~~ sign up thru Canvas

LaTeX not necessary, but must be easily readable

[Group of 1 is OK.]

(2) Breakout rooms:

- Assign randomly today
- Perhaps you pick out groups soon on Zoom
- Breakout problem available before class on course webpage

(3) Try to listen to tape lectures in a reasonable time frame;

Q: Recording of lectures are available through the Zoom like on Canvas CPSC 421/501 page.

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Return to:

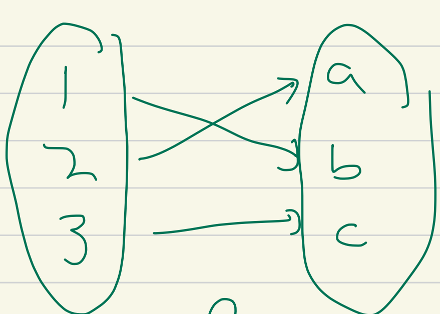
A set, $S$, is countably infinite if there is $S$ is the <u>same size</u> as $\mathbb{N} = \{1, 2, 3, \ldots\}$.

[ Recall: $S$ and $T$ have the <u>same</u> <u>size</u> if there is a bijection $S \to T$. ]

E.g.

$S$ $T$

"graphical representation"



$f$

"bijection"

"<u>one-to-one</u> <u>correspondence</u>"

"perfect matching"

We can ~~write~~ describe $f$ via

$$f(1) = b, \quad f(2) = a, \quad f(3) = c$$

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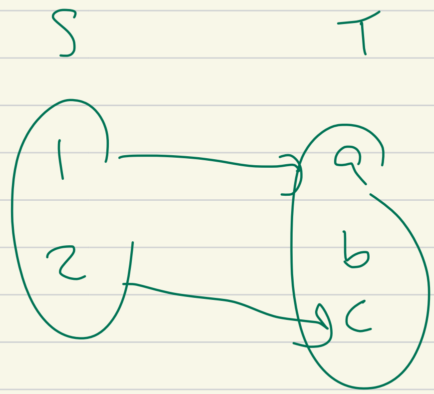
Injection
Injective map
"one-to-one"
into

looks like




If there is an injection $S \to T$,

then $|S| \leq |T|$.

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Example $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countable (even though $\mathbb{N} \underset{\text{proper}}{\subset} \mathbb{Z}$ )

$$\begin{array}{lcccccc}
\mathbb{Z}: & 0, & 1, & -1, & 2, & -2, & \dots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{N}: & 1, & 2, & 3, & 4, & 5, & \dots
\end{array}$$

Example: Positive rational numbers are countable (in textbook)

$$
\begin{array}{ccccc}
1/1 & 1/2 & 1/3 & 1/4 & 1/5 \ \cdots \\
2/1 & 2/2 & 2/3 & 2/4 & \cdots \\
3/1 & 3/2 & 3/3 & 3/4 & \cdots \\
4/1 & 4/2 & 4/3 & 4/4 & \cdots
\end{array}
$$

Gives

$$\frac{1}{1}, \quad \frac{1}{2}, \frac{2}{1}, \quad \frac{1}{3}, 2/2, 3/1, \ldots$$

already occurs

Sometimes it convenient not to have discard extra copies

$S$ is <u>countably infinite</u> if it has same size as $\mathbb{N}$.

$S$ is countable if either
(1) $S$ is finite, or
(2) $S$ is countably infinite.

<u>Thm</u>: $S$ is countable iff there is a <u>surjective</u> map $\mathbb{N} \to S$.

<u>Idea</u>: $S = \{1,2,3\}$

list elements of $S$ in sequence

$$\begin{array}{llllllll} S & 1, & 2, & 3, & 3, & 3, & 3, & \ldots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbb{N} & 1, & 2, & 3, & 4, & 5, & 6, & 7 \ldots \end{array}$$

surjection

$S$ is uncountable if $S$ is not countable,

i.e. there is no surjection $\mathbb{N} \to S$.

Example ①

Set of languages over a finite alphabet $\Lambda$

$= \text{Power}(\Lambda^*) = \{ \text{subsets of } \Lambda^* \}$

e.g. $\Lambda = \{0, 1, \ldots, 9\}$

Most important

$\text{PRIMES} = \{ 2, 3, 5, 7, 11, \ldots \}$

$\text{PRIMES} \in \text{Power}(\Lambda^*)$

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Example ② $\mathbb{R}$ = real numbers are uncountable.

Also $\Lambda^*$, for any ~~fin~~ alphabet $\Lambda$

(finite subset) is countable

$\Lambda = \{a, b, c\}$

$$\Lambda^* = \Lambda^0 \cup \Lambda^1 \cup \Lambda^2 \cup \ldots$$

$\varepsilon \qquad a, b, c \qquad aa, ab, ac, ba, bb, \ldots cc$

you can write all of $\Lambda^*$ as a sequence, equivalently $\Lambda^*$ and $\mathbb{N}$ has the same size.

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algorithm ~~Subset~~ subset of (say) ASCII$^*$

or $\Lambda^*$

# Set of languages is uncountable.

Fundamental tool to prove a language is uncountable is Cantor's theorem.

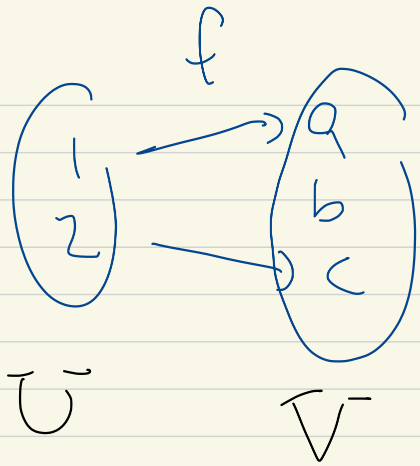
Cantor's Theorem: If $S$ is a set, and $f : S \to \text{Power}(S)$, then $f$ is not surjective; moreover

$$T = \{ s \in S \mid s \notin f(s) \}$$

then $T$ is not equal to $f(t)$ for any $t \in S$; i.e. $T$ is not in the image of $f$

[The image of a map $g : U \to V$ are those $v \in V$ s.t. $v = g(u)$ for some $u \in U$ ]

e.g.,




then image$(f) = \{a, c\}$

b is not in the image

$b \neq f(1)$, $b \neq f(2)$

$v \in V$ s.t. $v = g(u)$ for some $U$

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Say EXAMPLE:

$f : \{1,2,3\} \to \text{Power}(\{1,2,3\})$

$\underbrace{\{1,2,3\}}$

$[3] \to \text{Power}([3])$

s.t.

$f(1) = \{1,2\}$, $f(2) = \emptyset$, $f(3) = [3]$

$= \{1,2,3\}$

$1 \in f(1)$

$2 \notin f(2)$

$3 \in f(3)$

$T = \{2\}$

1 doesn't

$T \neq f(1)$ $\quad$ $1 \in f(1)$

$\qquad\quad \shortparallel$

$\quad \{1,2\}$ $\quad$ $1 \notin T$

$T \neq f(2)$ $\quad$ $2 \notin \emptyset$

$\qquad\quad \shortparallel$

$\quad \emptyset$ $\quad$ $2 \in T$

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Proof: If $T = f(t)$ for some $t \in S$

Either ① $t \in T$ or ② $t \notin T$,

but both ① & ② are impossible!

① $t \in T = \{ s \in S \mid s \notin f(s) \}$, $t \notin f(t)$

$$\begin{matrix} t \notin f(t) \\ \shortparallel \\ T \end{matrix}$$

② similarly impossib $\quad$ $t \notin T$

$$\text{Power}([3]) = \{ \emptyset, \{1\}, \{2\}, \{1,2\}, \ldots \underbrace{\{1,2,3\}}_{}\}$$

$$\{1,2,3\} = [3]$$

$[3] = \{1,2,3\}$ notation

Cor: If $S$ is countably infinite then $\underline{\text{Power}(S)}$ is $\underline{\text{uncountable}}$.

$$\mathbb{N} \overset{\text{bij}}{\longleftrightarrow} S$$

if Power(S) were countable then you have map

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\text{surj}"] \arrow[d, leftrightarrow] & \text{Power}(S) \\
S \arrow[ur, dashed] &
\end{tikzcd}$$

Breakout rooms: Pick one or two of problems (1), (3), (5) above

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(5) maps to help remember $\begin{cases} \text{injective} \\ \text{surjective} \\ \text{bijective} \end{cases}$

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Convince yourself that $T$ is not in the image of $f$. [This is not a precise task.]

$$\mathbb{N} \times \mathbb{N} = \mathbb{N}^2 = \{(a,b) \mid a,b \in \mathbb{N}\}$$

$$\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Idea

$1,1 \to 11$ $\quad$ $101$

$1,2 \to 12$ $\quad$ $102$

$15,3 \to 153$ $\quad$ $1503$

$1, 53 \to 153 \quad 1053$

$\underbrace{\qquad\qquad}_{\text{order by } <}$

101, 102, 201, ...

(1,1)

(a,b) look at

at

(1,2) (2,1)

(1,3) (2,2) (3,1)

⋮

1) Describe in words:

Describe

$(1,1) \longleftrightarrow 1$

$(1,2) \longleftrightarrow 2$

$(2,1) \longleftrightarrow 3$

if the rank of $(a,b)$ is $a+b$, then...

2 Give a formula