Topics for today or soon:

Section 4 of handout:

- Countable Sets, Uncountable Sets
- Cantor's Theorem

Sections 5 and 6

- Russell's Paradox and Set Theory Subtleties
- Related Paradoxes and Theorems
BREAKOUT ROOM PROBLEMS

① Show that $\mathbb{N} \times \mathbb{N}$ is countable

② Show that $\mathbb{Z} \times \mathbb{Z}$ is countable

③ Let $f : \{1, 2, 3, 4\} \to \text{Power}(\{1, 2, 3, 4\})$ (where $\{1, 2, 3, 4\}$ be given by:

$f(1) = \emptyset, \ f(2) = \{1, 2\}, \ f(3) = \{2, 4\}, \ f(4) = \{1\}$,

Describe $T = \{ s \in \{1, 2, 3, 4\} \mid s \notin f(s) \}$.

Convince yourself that $T$ is not in the image of $f$. [This is not a precise task.]
4) Is $\text{Power}(\{a,b\}^*) \cup \mathbb{N}$ countable?

5) Find an example of an injective and/or surjective and/or bijective map to help you remember these terms.

(This is not a precise question.)

E.g., $\{\text{students}\} \rightarrow \{\text{ID numbers}\}$
Admin Stuff!

1. Individual Homework!  
   - Write up individually
   - Say whom you worked with

2. Group Homework!  
   - Submit as a group
   - Same grade as your group members
   - You can work in groups of up to 4

Gradescope sign up thru Canvas

LaTeX not necessary, but must be easily readable

[Group of 1 is OK.]

2. Breakout rooms:

   - Assign randomly today
   - Perhaps you pick out groups soon on Zoom
   - Breakout problem available before class
     on course webpage

3. Try to listen to tape lectures in a
   reasonable time frame:
Recording of lectures are available through the Zoom like on Canvas CPSC 421/501 page.

Return to:

A set, \( S \), is countably infinite if there is \( S \) is the same size as \( \mathbb{N} = \{ 1, 2, 3, \ldots \} \).

Recall: \( S \) and \( T \) have the same size if there is a bijection \( S \rightarrow T \).

E.g., \( S \rightarrow T \)

"graphical representation" \[
\begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a & b & c \\
\end{pmatrix}
\]

"bijection" "one-to-one correspondence" "perfect matching"
We can define $f$ via
\[
\begin{align*}
  f(1) &= b, \\  f(2) &= a, \\  f(3) &= c
\end{align*}
\]

An injection is an injective map, often described as "one-to-one" (with no two elements of $S$ mapping to the same element of $T$).

If there is an injection $S \to T$, then $|S| \leq |T|$.

Example: $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ is countable (even though $\mathbb{N} \subset \mathbb{Z}$).

$\mathbb{Z} : 0, 1, -1, 2, -2, \ldots$

$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$

$\mathbb{N} : 1, 2, 3, 4, 5, \ldots$
Example: Positive rational numbers are countable (in textbook)

\[ \frac{1}{1} \rightarrow \frac{1}{2} \rightarrow \frac{1}{3} \rightarrow \frac{1}{4} \rightarrow \cdots \]

\[ 2 \rightarrow 2 \rightarrow 2 \rightarrow \frac{1}{3} \rightarrow \frac{1}{4} \rightarrow \cdots \]

\[ 3 \rightarrow 3 \rightarrow 3 \rightarrow 3 \rightarrow 3 \rightarrow \cdots \]

\[ 4 \rightarrow 4 \rightarrow 4 \rightarrow 4 \rightarrow 4 \rightarrow \cdots \]

Gives

\[ \frac{1}{1}, \quad \frac{1}{2}, \quad \frac{2}{1}, \quad \frac{1}{3}, \quad \frac{2}{3}, \quad \frac{3}{1}, \quad \cdots \]

Sometimes it convenient

not to have discard extra
$S$ is countably infinite if it has some size as $\mathbb{N}$.

$S$ is countable if either

$\text{1. } S$ is finite, or

$\text{2. } S$ is countably infinite.

Thm: $S$ is countable iff there is a surjective map $\mathbb{N} \to S$.

Idea: $S = \{1, 2, 3\}$

List elements of $S$ in sequence

$S$: 1, 2, 3, 3, 3, 3, ...  

$\mathbb{N}$: 1, 2, 3, 4, 5, 6, ...  

\[\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \ 	ext{surjection}\]
$S$ is uncountable if $S$ is not countable, i.e., there is no surjection $\mathbb{N} \rightarrow S$.

Example (i)

Set of languages over a finite alphabet $\Lambda$:

$$\text{Power} (\Lambda^*) = \{ \text{subsets of } \Lambda^* \}$$

E.g., $\Lambda = \{0, 1\}$, $\Lambda^* = \{0, 1\}^*$

Most importantly:

$$\text{PRIMES} = \{ 2, 3, 5, 7, 11, \ldots \}$$

$$\text{PRIMES} \subseteq \text{Power} (\Lambda^*)$$

Example (2) $\mathbb{R}$: real numbers are uncountable.
Also \( \Lambda^* \), for any alphabet \( \Lambda \)

(finite subset) is countable

\[ \Lambda = \{ a, b, c \} \]

\[ \Lambda^* = \Lambda^0 \cup \Lambda^1 \cup \Lambda^2 \cup \ldots \]

you can write all of \( \Lambda^* \) as a sequence, equivalently \( \Lambda^* \) and \( \mathbb{N} \)

has the same size.

algorithm subset of (say) ASCII* or \( \Lambda^* \)
Set of languages is uncountable.

Fundamental tool to prove a language is uncountable is Cantor’s theorem.

Cantor’s Theorem: If \( S \) is a set, and \( f: S \rightarrow \text{Power}(S) \), then \( f \) is not surjective; moreover

\[
T = \{ s \in S \mid s \notin f(s) \}
\]

then \( T \) is not equal to \( f(t) \) for any \( t \in S \), i.e. \( T \) is not in the image of \( f \).

The image of a map \( g: U \rightarrow V \) are those \( v \in V \) s.t. \( v = g(u) \) for some \( u \in U \).
e.g., \[
\begin{array}{ccc}
\text{graph} & f & \text{then image}(f) \\
\begin{array}{c}
1 \\
2
\end{array} & \begin{array}{c}
a \\
b \\
c
\end{array} \\
\end{array}
\]
\[
\text{image}(f) = \{a, c\} \\
U \lor \lor b \text{ is not in the image}
\]

\[
b \neq f(1), \ b \neq f(2)
\]

\[
v \in V \text{ s.t. } v = g(w) \text{ for some } w
\]

Say \[
f : \{1, 2, 3\} \to \text{Power}(\{1, 2, 3\})
\]

\[
\begin{array}{c}
\text{EXAMPLE:} \\
\text{s.t.}
\end{array}
\]

\[
f(1) = \{1, 2\}, \ f(2) = \emptyset, \ f(3) = [3]
\]

\[
1 \in f(1), \ 2 \notin f(2), \ 3 \in f(3)
\]

\[
T = \{2\}
\]

\[
1 \text{ doesn't belong}
\]
Proof: If $T = f(t)$ for some $t \in S$

Either (1) $t \in T$ or (2) $t \not\in T$,
but both (1) & (2) are impossible.

(1) $t \in T = \{ s \in S \mid s \in f(s) \}$, $t \not\in f(4)$

(2) similarly impossible

$\text{Power}(\{1,2\}) = \{ \emptyset, \{1\}, \{2\}, \{1,2\}, \ldots \}$
\[3\] = \{1, 2, 3\} \text{ notation}

Cor: If \( S \) is countable, infinite then \( \text{Power}(S) \) is uncountable.

\[\begin{align*}
\mathbb{N} & \leftrightarrow S \\
\text{if} \ \text{Power}(S) & \text{ were countable} \\
\text{then you have map} \\
\mathbb{N} & \twoheadrightarrow \text{Power}(S) \\
S & \rightarrow
\end{align*}\]
Breakout rooms: Pick one or two of problems ①, ③, ⑤ above

⑤ maps to help remember \{injective, surjective, bijective\}

① Show that $\mathbb{N} \times \mathbb{N}$ is countable.

③ Let $f : [4] \to \text{Power}([4])$ (where $[4] = \{1, 2, 3, 4\}$) be given by:

$f(1) = \emptyset$, $f(2) = \{1, 2\}$, $f(3) = \{2, 4\}$, $f(4) = [4]$.

Describe $T = \{ s \in [4] \mid s \notin f(s) \}$.

Convince yourself that $T$ is not in the image of $f$. [This is not a precise task.]
\( \mathbb{N} \times \mathbb{N} = \mathbb{N}^2 = \{ (a, b) \mid a, b \in \mathbb{N} \} \)

\[
(1,1) \quad (1,2) \quad (1,3) \\
(2,1) \quad (2,2) \quad (2,3) \\
(3,1) \quad (3,2) \quad (3,3)
\]

Idea 1,1 \rightarrow 111

1,2 \rightarrow 112

1,3 \rightarrow 153
153 → 153

order by

101, 102, 201

\((1,1)\) (a,b) look at at

\((1,2)\) (2,1)

\((1,3)\) (2,2) (3,1)

\(\vdots\)

\(\vdots\)

(1) Describe in words.
Describe

\[(1, 1) \rightarrow 1\]
\[(1, 2) \rightarrow 2\]
\[(2, 1) \rightarrow 3\]

if the rank of \((a, b)\) is \(a \neq b\), then...

Give a formula